

## Modeling of propagation of optical signals in gradient index media based on fractional Fourier transform

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**Abstract.** This research has simulated the propagation of the light beams through the quadratic index media. Five methods of simulation were considered and the propagating beams corresponding to different input signals such as Airy-Gaussian beams, rectangular function pulses, triangular function pulses, cosine function signals and finite eigenfunctions were demonstrated.

**Keywords:** gradient index media, fractional Fourier transform, Airy-Gaussian beams, Hermite-Gaussian modes, eigenfunctions

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### Introduction

Fractional Fourier transform (FrFT) is a set of linear transformations that generalize Fourier transform. Fourier transform is generally interpreted as a convention of the time domain of the signal to its frequency domain.

The canonical FrFT was considered [1] as the Fourier transform of  $\alpha$ -order, where  $\alpha$  is the real value. We can likewise define the FrFT as the operation of the frequency-time distribution (Wigner distribution function) rotation at a certain angle [2].

Originally, FrFT was used in quantum mechanics; however, recently it has increasingly become a focus of opticists. As a result, extensive research involving its properties, optical realization and potentiality opportunities in optic applications has been performed. Thus, currently FrFT is actively used in optical image processing [3]. Moreover, the fractioning of some transformation provides a new degree of freedom (fraction order) that can be used for a more complete description of the object (signal) or as an additional encoding parameter.

FrFT is used in differential equation solving, in quantum mechanics and quantum optics, in optical theory of diffraction, in optical system and optical signal processing

descriptions including the application of frequency filters, time filtration and multiplexing, as well as in pattern recognition, in wavelet-transformations, in operations with chirp-functions, in encryption, for neural network creation and other applications. A more detailed review of FrFT can be found in the paper [3] produced by T. Alieva et al.

The modular lens system and the system of several spherical and/or cylindrical lenses are among the methods of FrFT optical realization [4-8]. Some of these systems (especially cylindrical lenses) are used for astigmatic transformation in order to form vortex beams [5, 8-10].

One of application of FrFT is the description of laser beam propagation in gradient index media [11, 12].

In this work, we use one-dimensional FrFT to model optical signal propagation in optical waveguides with parabolic dependence of the refractive index. The eigenfunctions of the transforms are Hermite-Gaussian modes [1].

During the modelling process special attention is given to Airy-Gaussian beams, which carry finite power and keep the properties of non-diffracted propagation in a partial area. They can be experimentally realized with a particularly good approximation [13-16].

## 1. General theory

The light beam propagation through the ABCD-system in one-dimensional cases is described by the Huygens integral:

$$U_2(x_2) = \sqrt{\frac{k}{i2\pi B}} \int_{-\infty}^{+\infty} U_1(x_1) \exp\left[\frac{ik}{2B}(Ax_1^2 - 2x_1x_2 + Dx_2^2)\right] dx_1. \quad (1)$$

where  $k = 2\pi/\lambda$ ,  $U_1(x_1)$  is input field,  $U_2(x_2)$  is output field.

In gradient index media with the refraction index  $n = n_0(1 - x^2/2a^2)$ , the matrix of ABCD-system is (beam propagation from  $z_1 = 0$  to  $z_2 = z$ ) [13, 14, 17, 18]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(z/a) & a \sin(z/a) \\ -\sin(z/a)/a & \cos(z/a) \end{bmatrix}. \quad (2)$$

With this type of matrix the integral (1) turns into FrFT.

The complete set of FrFT eigenfunctions is the following set of Hermite-Gaussian functions:

$$\begin{aligned} \mathfrak{F}^\alpha[\psi_n(x)] &= e^{-i\alpha n\pi/2} \psi_n(x), \\ \psi_n(x) &= \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}x) e^{-\pi x^2}, \end{aligned} \quad (3)$$

where  $H_n(x)$  – a Hermite polynomial of order  $n$ :

$$H_n(x) = (-1)^n e^{x^2} \frac{d}{dx^n} e^{-x^2}. \quad (4)$$

If the FrFT has finite integration limits (in other words the input beam is limited), its eigenfunctions are somewhat different from the Hermite-Gaussian modes [19].

The propagation of a light beam through the gradient index media was modeled.

The Airy-Gaussian beams are:

$$U_1(x_1; \kappa_1, \delta_1, S_1, q_1) = Ai\left(\frac{x_1 + \delta_1}{\kappa_1}\right) \exp\left[iS_1\left(\frac{x_1 + \delta_1}{\kappa_1}\right) + i\frac{S_1^2}{3} + \frac{ikx_1^2}{2q_1}\right], \quad (5)$$

$$\kappa_1, \delta_1, S_1, q_1 \in C.$$

Furthermore, the cosine distribution function is:

$$f(x) = A \cos(\omega x + \varphi), \quad (6)$$

$$A, \omega, \varphi \in R.$$

In addition, the rectangular function, triangle function and eigenfunctions have been simulated.

## 2. Simulation

Method 1: direct numerical calculation of FrFT by definition.

The integral (1) is rarely solved analytically, so we use numerical computations. If typical methods are used, the numerical calculations of quadratic exponents require a very large number of sampling points because of rapid oscillations in the kernel. The problem is especially pronounced if  $\alpha$  is close to 0 or  $\pm 2$ . We assume that functions and their Fourier transformations are limited (they are not equal to zero in finite intervals) and this difficulty can be avoided. If  $0.5 \leq \alpha \leq 1.5$  or  $2.5 \leq \alpha \leq 3.5$ , we can directly calculate the integral. If  $-0.5 < \alpha < 0.5$  or  $1.5 < \alpha < 2.5$ , we can use the property of additivity:  $\mathfrak{F}^\alpha = \mathfrak{F}^1 \mathfrak{F}^{\alpha-1}$ , where in the transformation of  $\alpha-1$  order may be found immediately.

For the integration we use Simpson's rule ( $n$  is even):

$$\int_a^b f(x) dx \approx \frac{h}{3} \sum_{j=1}^{n/2} (f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})), \quad (7)$$

$$x_j = a + jh, \quad j = \overline{0, n-1}, \quad h = (b-a)/n.$$

Although this method of FrFT calculation can produce accurate results, it operates slowly and has a computational complexity  $O(N^2)$  [20].

Method 2: Fast FrFT.

FrFT is a special case of the more general transform class sometimes known as linear canonical transformations or quadratic-phase transformations. The elements of this class can be decomposed to a sequence of simple operations such as chirp-multiplication, chirp-convolution, scaling and typical Fourier transform. There are two different decompositions demonstrated here, leading to different algorithms.

Method 3: the decomposition into chirp-multiplication, chirp-convolution and another single chirp-multiplication sequence.

In this approach, we assume that  $-1 \leq \alpha \leq 1$ . The transformation (1) can be written as:

$$f_\alpha(x) = \exp\left[-\frac{i\pi}{\lambda a} x^2 \tan\left(\frac{\varphi}{2}\right)\right] g'(x), \quad (8)$$

$$g'(x) = A_\varphi \int_{-\infty}^{+\infty} \exp\left[\frac{i\pi}{\lambda a} \beta (x-x')^2\right] g(x') dx', \quad (9)$$

$$g(x) = \exp\left[-\frac{i\pi}{\lambda a} x^2 \tan\left(\frac{\varphi}{2}\right)\right] f(x), \quad (10)$$

where  $g(x), g'(x)$  represents an intermediate result,  $\varphi = \frac{z}{a}$ ,  $\beta = \csc \varphi$ ,  $A_\varphi = \sqrt{\frac{\beta}{i\lambda a}}$ .

Method 4: another decomposition involving the Whittaker-Shannon interpolation formula (sinc-interpolation). The defining expression of FrFT can be written as:

$$\begin{aligned} \{\mathfrak{S}^\alpha f\}(x) &= A_\varphi \exp\left(\frac{i\pi}{\lambda a} \alpha x^2\right) \times \\ &\times \int_{-\infty}^{+\infty} \exp\left(-\frac{i2\pi\beta}{\lambda a} xx'\right) \exp\left(\frac{i\pi}{\lambda a} \alpha x'^2\right) f(x') dx'. \end{aligned} \quad (11)$$

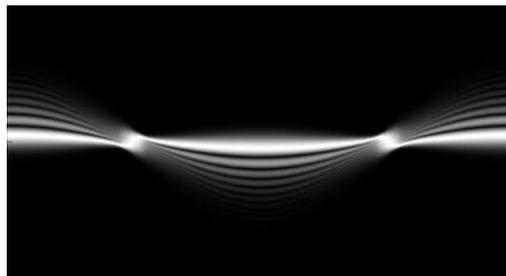
The function  $\exp\left(\frac{i\pi}{\lambda a} \alpha x'^2\right) f(x')$  can be represented by the Shannon interpolation formula:

$$\begin{aligned} \exp\left(i\pi\alpha x'^2\right) f(x') &= \sum_{-N}^N \exp\left[\frac{i\pi}{\lambda a} \alpha \left(\frac{n}{2\Delta x}\right)^2\right] \times \\ &\times f\left(\frac{n}{2\Delta x}\right) \text{sinc}\left(2\Delta x\left(x' - \frac{n}{2\Delta x}\right)\right). \end{aligned} \quad (12)$$

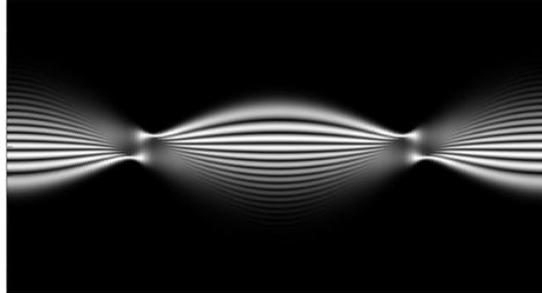
We substitute (12) in (11), change sequence of integration and summation and use certain algebraic manipulations to obtain the following expression:

$$\begin{aligned} \{\mathfrak{S}^\alpha f\}\left(\frac{m}{2\Delta x}\right) &= \frac{A_\varphi}{2\Delta x} \exp\left[\frac{i\pi}{\lambda a} (\alpha - \beta) \left(\frac{m}{2\Delta x}\right)^2\right] \times \\ &\times \sum_{-N}^N \exp\left[\frac{i\pi}{\lambda a} \beta \left(\frac{m-n}{2\Delta x}\right)^2 + \frac{i\pi}{\lambda a} (\alpha - \beta) \left(\frac{n}{2\Delta x}\right)^2\right] f\left(\frac{n}{2\Delta x}\right). \end{aligned} \quad (13)$$

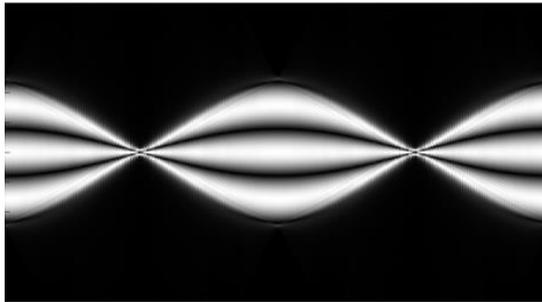
There are other methods, for example those described in [21], however they are not well suited for plotting images on a plane. The modeling results are demonstrated in Figures 1-6.



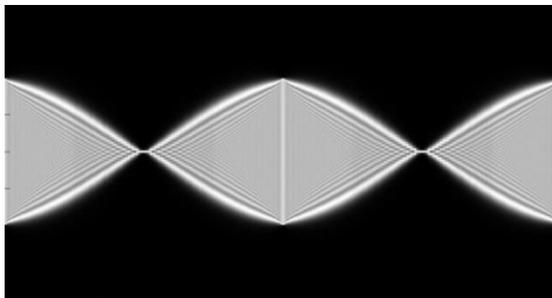
**Fig. 1.** – The propagation of Airy-Gaussian beams (version 1)



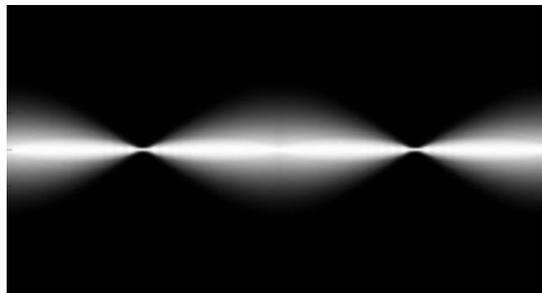
**Fig. 2.** – The propagation of Airy-Gaussian beams (version 2)



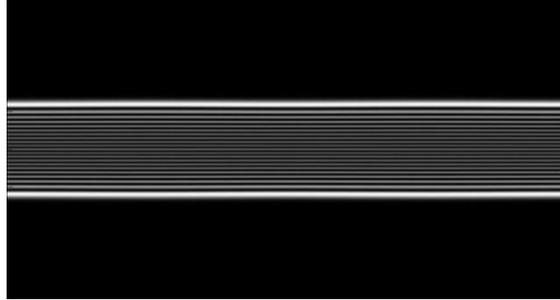
**Fig. 3.** – The propagation of cosine function signals



**Fig. 4.** – The propagation of rectangular pulses



**Fig. 5.** – The propagation of triangular pulses



**Fig. 6.** – The propagation of eigenfunctions

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