Conceptual Methods for Identifying Needs of Mobile Network Subscribers

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Abstract. The paper is devoted to methods for identifying payment plans and services by mobile operators which are the best for the given subscribers. We base our research on the model-theoretic approach to domain formalization. We use Formal Concept Analysis for processing the mobile subscriber data. An Ontological Model of the domain "Mobile Networks" is constructed in the scope of this research. The Ontological Model of the domain is constructed by integration of data extracted from depersonalized subscriber profiles. The signature of this Ontological Model contains unary predicates which describe subscriber behavior and features of payment plans and services. We consider formal contexts where objects are subscriber models and attributes are formulas of predicate logic. We investigate concept lattices and association rules of these formal contexts. Knowledge about optimal payment plans and services for a given subscriber is generated automatically with the help of the association rules.

Keywords: mobile networks, mobile network subscribers, formal context, concept lattice of formal context, association rules, ontology, ontological model.

1 Introduction

Mobile connection is a very important part of our life. Mobile operators provide the possibility to be in touch for people in different countries. Operators provide access to USSD-applications and to the Internet.

Mobile operators develop various payment plans and services to satisfy their clients' needs. However it is difficult for mobile network subscribers to get up-to-date information about new payment plans or services. Mobile operators send SMS messages to inform clients about news. But it is very expensive to inform all subscribers about every small change or update of services. A possible solution of this problem is sending personal recommendations about services and payment plans that could be useful for a given subscriber.

A visualization approach based on a graph of calls made by subscribers was used in [1] for mining behavior patterns of mobile network subscribers. A behavior pattern discovered during the graph exploration resulted in developing and applying a new payment plan. Development of methods for increasing the number of subscribers using services by a mobile network is studied in [2]. An algorithm called Frequent Pattern-Growth Strategy is used for mining patterns in how subscribers use mobile network services. Optimization strategies are suggested by experts based on series of ‘frequent’ sets.

Formal Concept Analysis is a well-known formalism in data analysis and knowledge engineering, see recent surveys [3, 4]. Formal Concept Analysis is used to develop user behavior templates [5, 6]. These results are applied to planning and running marketing campaigns.

Association rules for optimizing structures of menus for accessing mobile network services were constructed in [7]. The Apriori algorithm was used in [8] to develop association rules patterns in services visited during a single subscriber session. Today we have more effective algorithms for mining association rules, e.g. see [9].

Fuzzy concept lattices were first introduced in [10]. Papers [11-13] are devoted to definitions of fuzzy transaction, support and confidence of fuzzy association rules. The authors of [11] used an algorithm developed in [14] for building sets of fuzzy rules which describe dependencies between popular telecom services provided by mobile networks in Taiwan.

Our research is devoted to methods for identifying payment plans and services which would be optimal for a given mobile network subscriber. Such knowledge allows mobile operator to make really useful recommendations for subscribers.

We base our research on the model-theoretic approach to domain formalization [15-18]. We use methods and techniques of Formal Concept Analysis for processing the mobile subscriber data. Now a lot of attention is paid to the relationships between FCA and models of knowledge representation and processing [19].

The ontological model of the domain “Mobile Networks” is constructed by integration of data extracted from depersonalized subscriber profiles. The signature of this ontological model contains unary predicates which describe subscriber behavior and features of payment plans and services. To generate meaningful recommendation of alternative services and payment plans, we define formal contexts where objects are subscriber models, and attributes are formulas of predicate logic. We investigate concept lattices and association rules of these formal contexts to get high-quality recommendation. To do this, we consider extensions of attribute sets of formal contexts.

In [20] extensions of infinite attribute sets were considered, it was suggested to use concept descriptions of bounded depth. In [21] a new approach to reduce the number of attributes was presented.

In this paper we consider finite extensions of the initial finite context. We use interrelation between axiomatizable classes and FCA [22]. Section 2.1 is devoted to isomorphisms between lattices of relatively axiomatizable classes of one-element models and lattices of formal concepts of formal contexts generated by these classes. Section 2.2 describes extensions of such formal contexts having distributive concept lattices.

The main purpose of this paper is to develop methods of identifying payment plans and services which would be optimal for the given mobile network subscriber. To do this, firstly, we construct Case Model based on the known information about behavior patterns of mobile network subscribers (Section 2.2). We represent the Case Model as a relatively axiomatizable class of one-element models. On the base of this Case Model we define a formal context.

Secondly, we move from the Case Model to Ontology Model (Section 3.1). We construct the set of ontological projections which is the basis of extensions of attribute set of the formal context under consideration (Section 3.2).

And finally we mine association rules with high confidence and support in the extended formal context. Computer experiments show that the methods presented in the paper allow us to find association rules which can be used for recommendations.

2 Case Model

2.1 Relatively axiomatizable classes and formal contexts

Here we introduce some definitions and results on the relationship between relatively axiomatizable classes and formal contexts. The main result of this section is Proposition 2 which is necessary for proofs of Propositions 4 and 5 in Section 2.2. The proofs of the statements are based on [22].

An algebraic system (a model) is a tuple $\mathfrak{A} = \langle A;\ P_1,\ldots,P_n, f_1,\ldots,f_m, c_1,\ldots,c_k\rangle$, where the set $|\mathfrak{A}| = A$ is called universe, $P_1,\ldots,P_n$ are predicates defined on the set A, $f_1,\ldots,f_m$ are functions defined on the set A and $c_1,\ldots,c_k$ are constants. The tuple $\sigma = \langle P_1,\ldots,P_n, f_1,\ldots,f_m, c_1,\ldots,c_k\rangle$ is called signature of the algebraic system $\mathfrak{A}$.

Denote by $FV(\varphi)$ the set of all free variables of a formula φ. A formula having no free variables is called sentence. For a signature σ we denote:

$F(\sigma) \leftrightharpoons \{\varphi \mid \varphi\ is\ a\ formula\ of\ the\ signature\ \sigma\}$,

$F_1(\sigma) \leftrightharpoons \{\varphi \mid \varphi \in F(\sigma)\ and\ FV(\varphi) = \{x\}\}$,

$S(\sigma) \leftrightharpoons \{\varphi \mid \varphi\ is\ a\ sentence\ of\ the\ signature\ \sigma\}$ and

$K(\sigma) \leftrightharpoons \{\mathfrak{A} \mid \mathfrak{A}\ is\ a\ model\ of\ the\ signature\ \sigma\}$.

Here $FV(\varphi) = \{x\}$ means that each formula $\varphi \in F_1(\sigma)$ has just one free variable, which is the fixed variable x.

Consider a signature σ and a model $\mathfrak{A} \in K(\sigma)$. For a sentence $\psi \in S(\sigma)$ we denote $\mathfrak{A} \vDash \psi$ if ψ is true in the model $\mathfrak{A}$. For a formula $\varphi(x_1,\ldots,x_n) \in F(\sigma)$ we write $\mathfrak{A} \vDash \varphi$ if $\mathfrak{A} \vDash \forall x_1 \ldots \forall x_n \varphi(x_1,\ldots,x_n)$.

Definition 1. *Let $K \subseteq K(\sigma)$. For a formula $\varphi \in F(\sigma)$ we denote $K \vDash \varphi$ if $\mathfrak{A} \vDash \varphi$ for any $\mathfrak{A} \in K$. For a set of formulas $\Gamma \subseteq F(\sigma)$ we denote $K \vDash \Gamma$ if $\mathfrak{A} \vDash \varphi$ for any $\mathfrak{A} \in K$ and $\varphi \in \Gamma$. For a set of formulas $\Gamma \subseteq F(\sigma)$ we denote*

$$K(\Gamma) \leftrightharpoons K_\sigma(\Gamma) \leftrightharpoons \{\mathfrak{A} \in K(\sigma) | \mathfrak{A} \vDash \varphi\ \ for\ \ any\ \varphi \in \Gamma\ \}.$$

A class $K \subseteq K(\sigma)$ is called axiomatizable if there exists a set $\Gamma \subseteq S(\sigma)$ such that $K = \{\ \mathfrak{A} \in K(\sigma) \mid \mathfrak{A} \vDash \Gamma\}$.

For the aims of our research we need to generalize the notion of relatively axiomatizable class [22] to the case of arbitrary sets of formulas Δ.

Definition 2. *Let $K, K_1 \subseteq K(\sigma)$ and $\Delta \subseteq F(\sigma)$. We say that the class K_1 is axiomatizable in the class K relatively to the set of formulas Δ if there exists a set $\Gamma \subseteq \Delta$ such that $K_1 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma\}$.*

Notice that the class $K_1 \subseteq K(\sigma)$ is axiomatizable if and only if K_1 is axiomatizable in the class $K = K(\sigma)$ relatively to the set of formulas $\Delta = S(\sigma)$.

Definition 3. *For $K \subseteq K(\sigma)$ and $\Delta \subseteq F(\sigma)$ we denote* $\mathbb{B}(K,\Delta) \leftrightharpoons \{K_1 \mid K_1 \textit{ is axiomatizable in } K \textit{ relatively to the set of formulas } \Delta\}$ *and* $T_\Delta(K) \leftrightharpoons \{\varphi \in \Delta \mid K \vDash \varphi\}$. *The set of formulas $T_\Delta(K)$ is call **Δ-type** of K.*

Note that $K_1 \in \mathbb{B}(K,\Delta)$ if and only if $K_1 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash T_\Delta(K_1)\}$.

For each class $K \subseteq K(\sigma)$ and set $\Delta \subseteq S(\sigma)$ we consider the formal context $(K, \Delta, \vDash)$, with derivation operator $()'$ [23].

Remark 1. *Let $K \subseteq K(\sigma)$, $\Delta \subseteq F(\sigma)$ and $A \subseteq K$. Then $A' = T_\Delta(K)$.*

For a formal context $(\mathrm{G}, \mathrm{M}, \mathrm{I})$ by $\underline{\mathfrak{B}}(\mathrm{G}, \mathrm{M}, \mathrm{I})$ we denote the lattice of formal concepts of the formal context $(\mathrm{G}, \mathrm{M}, \mathrm{I})$.

Proposition 1. *Let $K \subseteq K(\sigma)$, $\Delta \subseteq F(\sigma)$, $A \subseteq K$ and $B \subseteq \Delta$. Then $(A,B) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$ if and only if A is axiomatizable in the class K relatively to the set of formulas Δ and $B = T_\Delta(A)$.*

Proof. ($\Rightarrow$) Let $(A,B) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$. Then $B = A'$, so $B = T_\Delta(A)$ by Remark 1. We have $A = B'$, hence $A = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash B\}$ and $B \subseteq \Delta$. Therefore, by Definition 2, the class A is axiomatizable in the class K relatively to the set of formulas Δ.

($\Leftarrow$) Let the class A be axiomatizable in the class K relatively to the set of formulas Δ and $B \subseteq \Delta$. So there exists $\Gamma \subseteq \Delta$ such that $A = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma\}$. Then in the formal context $(K,\Delta,\vDash)$ we have $\Gamma' = A$. So $A'' = A$. The set $B = T_\Delta(A)$, thus $B = A'$ by Remark 1. Therefore, $(A,B) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$.

Corollary 1. *Let $K, K_1 \subseteq K(\sigma)$ and $\Delta \subseteq F(\sigma)$.*

1. $K_1 \in \mathbb{B}(K,\Delta)$ if and only if $(K_1, T_\Delta(K_1)) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$.

2. $K_1 = K_1''$ if and only if K_1 is axiomatizable in the class K relatively to the set of formulas Δ.

Therefore, the classes which are axiomatizable in a class K relatively to a set of formulas Δ are exactly extents of the formal concepts of the formal context $(K,\Delta,\vDash)$.

We consider $\mathbb{B}(K,\Delta)$ as a set ordered by inclusion $\subseteq$. So $\mathbb{B}(K,\Delta)$ is a lattice.

Proposition 2. *The lattices $\underline{\mathfrak{B}}(K,\Delta,\vDash)$ and $\mathbb{B}(K,\Delta)$ are isomorphic, i.e., $\underline{\mathfrak{B}}(K,\Delta,\vDash) \cong \mathbb{B}(K,\Delta)$, for any $K \subseteq K(\sigma)$ and $\Delta \subseteq F(\sigma)$.*

Proof. Let us consider the mapping $h\colon \underline{\mathfrak{B}}(K,\Delta,\vDash) \to \mathbb{B}(K,\Delta)$ defined as follows: $h(A,B) = A$ for any $(A,B) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$. By Proposition 1 for any $(A,B) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$ we have $h(A,B) = A \in \mathbb{B}(K,\Delta)$. For each $A \in \mathbb{B}(K,\Delta)$ it is true that $(A, T_\Delta(A)) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$, so $h(A, T_\Delta(A)) = A$. Thus the mapping h is onto.

For any $(A_1,B_1), (A_2,B_2) \in \underline{\mathfrak{B}}(K,\Delta,\vDash)$ we have: $(A_1,B_1) \le (A_2,B_2)$ iff $A_1 \subseteq A_2$. Hence the mapping h preserves the partial order.

Therefore, the mapping h is an isomorphism.

2.2 Description of the Case Model

Further we consider signatures consisting of a finite set of unary predicate symbols, i.e. $\sigma =< P_1, \dots, P_n >$. We consider the set $\Delta \subseteq S(\sigma)$ for different signatures σ which means that the original signature is enriched by new unary predicate symbols. From a model-theoretic point of view we may assume that there is some covering signature σ^U and all considered signatures are its subsets.

Consider a finite set $A = \{e_1, \dots, e_n\}$ of subscribers of a given mobile network and fix a signature $\sigma = \sigma_{\mathbb{P}} \cup \sigma_{\mathbb{Q}}$ where $\sigma_{\mathbb{P}}$ is a set of personal characteristics of subscribers and $\sigma_{\mathbb{Q}}$ is a set of payments plans, services and options. Each of these sets has a hierarchical structure. There are more details about the signatures $\sigma_{\mathbb{P}}$ and $\sigma_{\mathbb{Q}}$ below. For each subscriber e_i we know which characteristics (presented by signature predicates from σ) are true and which characteristics are false. Thus, for each subscriber e_i there is a one-element model $\boldsymbol{e_i} = \langle\{e_i\}, \sigma\rangle$ which is called a case of the domain $\mathbb{M}$.

Consider the ***Case Model*** $\mathfrak{A} = \langle A, \sigma\rangle$ defined by a set of cases $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ [20]. On the model $\mathfrak{A}$ for each signature predicate $P \in \sigma$ and for every element $e \in A$ we have $\mathfrak{A} \vDash P(e)$ if and only if the predicate $P(x)$ is true in the model (case) $\boldsymbol{e}$ (i.e., $\boldsymbol{e} \vDash P(x)$). Here $\boldsymbol{e} \vDash P(x)$ means that $\vDash \exists x P(x)$. On the base of the Case Model $\mathfrak{A} = \langle A, \sigma\rangle$ in the section 3.5 we will define the ontological model.

Denote by $K_{\mathfrak{A}} = \{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ the class of cases (one-element models) generated by the set of subscribers $\{e_1, \dots, e_n\}$.

Note that $K_{\mathfrak{A}} = \{\langle\{e\};\ \sigma\rangle \mid e \in A \ and\ \langle\{e\};\ \sigma\rangle \leq \mathfrak{A}\}$. Here the notation $\boldsymbol{e} = \langle\{e\};\ \sigma\rangle \leq \mathfrak{A}$ means that the model $\boldsymbol{e}$ is a submodel of the model $\mathfrak{A}$. Recall that in pure predicate signature each subset of a model is the universe of its submodel.

Here we consider different sets of formulas $\Delta \subseteq F_1(\sigma)$. In particular, we consider $\Delta_\sigma = \{P(x) \mid P \in \sigma\} \subseteq F_1(\sigma)$. Denote by $C_{\mathfrak{A}}^{\Delta} = (K_{\mathfrak{A}}, \Delta, \vDash)$ the formal context having the set of objects $K_{\mathfrak{A}}$, the set of attributes Δ and the incidence relation $\vDash$. Denote $C_{\mathfrak{A}}^{\sigma} = (K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$.

$\Delta \subseteq F_1(\sigma)$ is a set of properties of the cases $\boldsymbol{e} \in K_{\mathfrak{A}}$, which are definable by formulas of the signature σ. When we change the set Δ we change the set of attributes of the formal context keeping fixed the set of objects $K_{\mathfrak{A}}$. Reductions and expansions of formal contexts were studied in [25].

Let us consider two formal contexts $C_1 = (G, M_1, I)$ and $C_2 = (G, M_2, I)$. Suppose that $M_1 \subseteq M_2$, $A \subseteq G$ and $A = A''$ в C_1. Then $A = A''$ in C_2.

We define a mapping $i: \underline{\mathfrak{B}}(G, M_1, I) \to \underline{\mathfrak{B}}(G, M_2, I)$ as follows: $i(A, B_1) = (A, B_2)$, where $A \subseteq G$, $B_1 \subseteq M_1$, $B_2 \subseteq M_2$, $A = A''$, $B_1 = A'$ in the context C_1 and $B_2 = A'$ in the context C_2.

Remark 2. *The mapping* $i: \underline{\mathfrak{B}}(G, M_1, I) \to \underline{\mathfrak{B}}(G, M_2, I)$ *is an isomorphic embedding of the lattice* $\underline{\mathfrak{B}}(G, M_1, I)$ *into the lattice* $\underline{\mathfrak{B}}(G, M_2, I)$.

Next consider an arbitrary signature σ_0 and an arbitrary class $K_0 \subseteq K(\sigma_0)$.

Remark 3. *Let* $\Delta \subseteq F(\sigma_0)$ *and* $\varphi_1, \dots, \varphi_n \in F(\sigma_0)$. *Then the mapping*
$i: \underline{\mathfrak{B}}(K_0, \Delta, \vDash) \to \underline{\mathfrak{B}}(K_0, \Delta \cup \{\varphi_1 \& \dots \& \varphi_n\}, \vDash)$ *is an isomorphism of lattices.*

Corollary 2. *a) The sets of association rules of the formal contexts* $(K_0, \Delta, \vDash)$ *and* $(K_0, \Delta \cup \{\varphi_1 \& \dots \& \varphi_n\}, \vDash)$ *coincide up to the substitution of the formula* $(\varphi_1 \& \dots \& \varphi_n)$ *by the set* $\{\varphi_1, \dots, \varphi_n\}$.

b) The sets of attribute implications of the formal contexts $(K_0, \Delta, \vDash)$ *u* $(K_0, \Delta \cup \{\varphi_1 \& \dots \& \varphi_n\}, \vDash)$ *coincide up to the substitution of the formula* $(\varphi_1 \& \dots \& \varphi_n)$ *by the set* $\{\varphi_1, \dots, \varphi_n\}$.

Corollary 3. *If* $\Delta, \Delta_1 \subseteq F(\sigma_0)$, $\Delta \subseteq \Delta_1$ *and the set* $\Delta_1 \setminus \Delta$ *consists of some conjunctions of formulas from* Δ *then the sets of attribute implications as well as the sets of association rules of the formal contexts* $(K_0, \Delta, \vDash)$ *and* $(K_0, \Delta_1, \vDash)$ *coincide up to the substitution of the conjunctions from* $\Delta_1 \setminus \Delta$ *by the corresponding sets of formulas.*

Let us go back to the formal context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$.

Remark 4. *Let* $P_1, P_2 \in \sigma$. *Then the mapping* $i: \underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma, \vDash) \to \underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma \cup \{P_1(x) \vee P_2(x)\}, \vDash)$ *is an isomorphic embedding of lattices; in the general case this mapping is not an isomorphism. Moreover, in the general case* $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma, \vDash) \ncong \underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma \cup \{P_1(x) \vee P_2(x)\}, \vDash)$.

Corollary 4. *In the general case if we add a disjunction* $(P_1(x) \vee \dots \vee P_k(x))$ *to the set of formulas* Δ_σ, *where* $P_1, \dots, P_k \in \sigma$, *then the set of association rules of the formal context* $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ *will be changed.*

Denote $\Delta_\sigma^{\vee} = \Delta_\sigma \cup \{(P_1(x) \vee \dots \vee P_k(x)), \mid P_i \in \sigma\}$.

We will be adding disjunctions of signature predicates into the set Δ_σ for improving association rules based on an algorithm for subscribers' behavior prediction. It means that we will consider the set of formulas $\Delta_\sigma^{\vee}$ instead of the set of formulas Δ_σ and the formal context $(K_{\mathfrak{A}}, \Delta_\sigma^{\vee}, \vDash)$ instead of the formal context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$.

Definition 4. We say that a set of formulas $\Delta \subseteq F(\sigma_0)$ is closed under disjunction if $(\varphi \vee \psi) \in \Delta$ for any $\varphi, \psi \in \Delta$.

Proposition 3. *Let* $K \subseteq K_{\mathfrak{A}}$ *and* $\Delta \subseteq F(\sigma)$. *If the set of formulas* Δ *is closed under disjunction then the lattice* $\mathbb{B}(K, \Delta)$ *is distributive.*

Proof. Assume that $\Delta \subseteq F(\sigma)$ and $K_1, K_2 \in \mathbb{B}(K, \Delta)$. Then $K_1, K_2 \subseteq K$ and there exist $\Gamma_1, \Gamma_2 \in \Delta$ such that $K_1 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma_1\}$ and $K_2 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma_2\}$.

Denote $\Gamma_3 = \Gamma_1 \cup \Gamma_2$ and $\Gamma_4 = \{(\varphi \vee \psi) \mid \varphi \in \Gamma_1 \text{ } and \text{ } \psi \in \Gamma_2)\}$. Then $K_1 \cap K_2 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma_3\}$, hence $(K_1 \cap K_2) \in \mathbb{B}(K, \Delta)$.

Let $\mathfrak{A} \in K$. Then $\mathfrak{A}$ is a one-element model. Therefore for any $\varphi \in \Gamma_1$ and $\psi \in \Gamma_2$ we have: $\mathfrak{A} \vDash (\varphi \vee \psi) \Leftrightarrow \mathfrak{A} \vDash \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \vee \psi(x_1, \dots, x_n)) \Leftrightarrow$

$$\Leftrightarrow \mathfrak{A} \vDash \forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n) \text{ } or \text{ } \mathfrak{A} \vDash \forall x_1 \dots \forall x_n \psi(x_1, \dots, x_n) \Leftrightarrow$$

$$\Leftrightarrow \mathfrak{A} \vDash \varphi \text{ } or \text{ } \mathfrak{A} \vDash \psi, \text{ where } FV(\varphi \vee \psi) = \{x_1, \dots, x_n\}.$$

Assume that $\mathfrak{A} \in (K_1 \cup K_2)$, then $\mathfrak{A} \in K_1 \text{ } or \text{ } \mathfrak{A} \in K_2 \Rightarrow$

$$\Rightarrow (\mathfrak{A} \in K \text{ } and \text{ } \mathfrak{A} \vDash \Gamma_1) \text{ } or \text{ } (\mathfrak{A} \in K \text{ } and \text{ } \mathfrak{A} \vDash \Gamma_2) \Rightarrow$$

$$\Rightarrow \mathfrak{A} \in K \text{ } and \text{ } (\mathfrak{A} \vDash \Gamma_1 \text{ } or \text{ } \mathfrak{A} \vDash \Gamma_2) \Rightarrow \mathfrak{A} \in K \text{ } and \text{ } \mathfrak{A} \vDash \Gamma_4.$$

Next, suppose that $\mathfrak{A} \nvDash \Gamma_1$ and $\mathfrak{A} \nvDash \Gamma_2$. So there exist $\varphi \in \Gamma_1$ and $\psi \in \Gamma_2$ such that $\mathfrak{A} \nvDash \varphi$ and $\mathfrak{A} \nvDash \psi$. Then $\mathfrak{A} \nvDash (\varphi \vee \psi)$, so $\mathfrak{A} \nvDash \Gamma_4$.

Thus, if $\mathfrak{A} \vDash \Gamma_4$ then $(\mathfrak{A} \vDash \Gamma_1 \text{ } or \text{ } \mathfrak{A} \vDash \Gamma_2)$. Hence, if $\mathfrak{A} \in K$ and $\mathfrak{A} \vDash \Gamma_4$ then $\mathfrak{A} \in K_1$ or $\mathfrak{A} \in K_2$, so $\mathfrak{A} \in (K_1 \cup K_2)$.

Therefore, $K_1 \cup K_2 = \{\mathfrak{A} \in K \mid \mathfrak{A} \vDash \Gamma_4\}$ and $(K_1 \cup K_2) \in \mathbb{B}(K, \Delta)$.

We proved that $(K_1 \cap K_2), (K_1 \cup K_2) \in \mathbb{B}(K, \Delta)$ for any $K_1, K_2 \in \mathbb{B}(K, \Delta)$. Hence, the lattice $\mathbb{B}(K, \Delta)$ is distributive.

Proposition 4. *The lattice of formal concepts* $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma^\vee, \vDash)$ *is distributive.*

Proof: in virtue of Proposition 2 and Proposition 3.

However the initial formal context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ does not have this good property.

Remark 5. *In the general case the lattice of formal concepts* $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ *is not distributive. It means that there exists a class* $K_{\mathfrak{A}}$ *such that the lattice* $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ *is not distributive.*

Remark 6. *Let* $\Delta \subseteq F_1(\sigma)$, $\Delta_\sigma \subseteq \Delta$ *and the set* $\Delta \backslash \Delta_\sigma$ *consists of some conjunctions of formulas from* Δ_σ. *Then there exists a class* $K_{\mathfrak{A}}$ *such that the lattice* $(K_{\mathfrak{A}}, \Delta, \vDash)$ *is not distributive.*

For the set of all formulas the situation is better.

Proposition 5. *1) The lattice of formal concepts* $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, F_1(\sigma), \vDash)$ *is distributive.*
2) The lattice of formal concepts $\underline{\mathfrak{B}}(K_{\mathfrak{A}}, F(\sigma), \vDash)$ *is distributive.*

Proof: in virtue of Proposition 2 and Proposition 3.

Association rule mining for the original context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ does not produce a lot of rules with high confidence. A lot of various payment plans and services exist, and commonly more than one service can be useful for the subscriber. The service that will be preferred by the user depends on many factors. Some of these factors can change time to time. So we cannot detect such factors in scope of formal context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$ because the context is based on long users' history.

Moreover, mobile operator can suggest 2-3 possible services and the subscriber may select himself the most useful service. That is why it makes sense to add disjunctions of signature predicates to Δ_σ and use context $(K_{\mathfrak{A}}, \Delta_\sigma^\vee, \vDash)$ on next steps.

There are two problems with association rules that were mined using formal context $(K_{\mathfrak{A}}, \Delta_\sigma^\vee, \vDash)$. First of all some of rules have high confidence, but their conclusions are disjunctions of meaningfully nonrelated services. Such association rules could not be used for recommendations. It will be looking like spam for mobile network subscribers. So experts should process all rules and select only meaningful rules. Second, processing the whole formal context $(K_{\mathfrak{A}}, \Delta_\sigma^\vee, \vDash)$ is very laborious computational procedure.

To solve these problems we are moving from the Case Model $\mathfrak{A} = \langle A, \sigma \rangle$ to the Ontological Model $\langle \mathfrak{A}, T^a, T^s, T^f \rangle$ of the domain. We add new unary predicates to the signature σ to describe meaning of payment plans and services. Using new predicates (from the signature $\sigma_{\mathbb{R}}$) we generate automatically meaningful disjunctions of original predicates from the signature σ.

3 Ontological Model of the domain

3.1 Ontology

Ontological Model of the domain consists of four parts [15]:

(1) The domain ontology, i.e. description of the structure and the meaning of the domain concepts.

(2) General knowledge and domain regularities, sentences which are true for every case.

(3) The set of cases from the domain, that we consider in the given moment. This is empirical knowledge about the domain; the set of cases that we are looking at in this article is represented by the model $\mathfrak{A} = \langle A, \sigma \rangle$.

(4) Estimated and probabilistic knowledge: probabilistic and confidential estimates, fuzzy values of sentences [16].

In this section we describe construction of the domain ontology.

From a model-theoretic point of view the domain ontology construction consists of description of the signature and creation of a set of axioms that describe the meaning of the concepts of the domain [17, 18]. To define the signature $\sigma_{\mathbb{M}}$ of the domain $\mathbb{M}$ = "Mobile networks" we consider two sets of attributes: $\sigma_{\mathbb{P}}$, the set of individual subscriber's features and $\sigma_{\mathbb{Q}}$, the set of various payment plans and services.

The set of attributes $\sigma_{\mathbb{P}}$, "Individual subscribers' feature" consists of two parts: $\sigma_{\mathbb{P}_1}$, "payment plans" and $\sigma_{\mathbb{P}_2}$, "accrual". Every part $\sigma_{\mathbb{P}_i}$ $(i = \underline{1, 2})$ consists of two subparts, such as $\sigma_{\mathbb{P}_{i1}}$, "traffic (and accrual) without roaming inside operator network",…, $\sigma_{P_{i6}}$, "traffic (and accrual) in common roaming", … . Each of the listed signatures consists of more detailed categories, e.g., $\sigma_{P_{111}}$, "Traffic SMS without roaming inside operator network". Every category $\sigma_{P_{ijk}}$ contains finite number of signature symbols $P^1_{1ij}, \dots, P^n_{1ij}$. For example, $P^1_{111}(x)$ = "Traffic of SMS without roaming inside network for subscriber x is not more than 50 SMS in month" and $P^1_{112}(x)$ = "Traffic of SMS without roaming inside network for subscriber x is more than 50 SMS in month".

Signature $\sigma_{\mathbb{Q}}$ consists of two parts: $\sigma_{\mathbb{Q}_1}$ and $\sigma_{\mathbb{Q}_2}$. Part $\sigma_{\mathbb{Q}_1}$ is "payment plans", it has hierarchical structure and consists of symbols of unary predicates. Each unary predicate describes the presence or absence of connected payment plan for subscribers. Signature $\sigma_{\mathbb{Q}_2}$, "services and options", consists of symbols of unary predicates. Each unary predicate describes the presence or absence of connected service or option.

To describe the domain ontology, we define a finite set of ontological axioms $\mathcal{A}x_a \subseteq F_1(\sigma_{\mathbb{P}} \cup \sigma_{\mathbb{Q}})$. We introduce the following axioms.

Axioms of hyponym-hyperonym. Hierarchical structure of the signature $\sigma_{\mathbb{Q}_1}$ is represented by axioms such as:

$$(Q^n_{ijk}(x) \rightarrow Q^n_{ij}(x)) \ and \ (Q^n_{ij}(x) \rightarrow Q_i(x)).$$

Axioms of completeness. For each predicates inside every class $\sigma_{\mathbb{P}_{ijk}}$ and class $\sigma_{\mathbb{Q}_1}$ for a given subscriber there must be at least one true predicate. The schemes of such axioms are the following:

$$\vee\left\{P(x) \mid P(x) \in \sigma_{\mathbb{P}_{ijk}}\right\} \ and \ \vee\left\{P(x) \mid P(x) \in \ \sigma_{\mathbb{Q}_1}\right\}.$$

Axioms of including. For example, if payment plan t contains “more than 100 free SMS” then it contains “more 50 free SMS”. The schemes of such axioms are the following:

$$(Q_{ijk}^{n_1}(x) \rightarrow Q_{ijk}^{n_2}(x)), \text{ where } \ n_2 < n_1.$$

Next we construct an extension $(\sigma_{\mathbb{P}} \cup \sigma_{\mathbb{Q}})$ of the signature $\sigma_{\mathbb{M}}$ by additional unary predicates that describe properties of payment plans and interests of subscribers. For that we introduce two types of concepts:

1) Concepts $\sigma_{\mathbb{R}}$. This is a set of features for different payment plans, services, and options. For example, amount of free calls time, volume of SMS package or of Internet package and etc. With the help of $\sigma_{\mathbb{R}}$ we can give formal definition for payment plans and services, i.e., formal definition of predicates of the signature $\sigma_{\mathbb{Q}}$.

2) Concepts $\sigma_{\mathbb{I}}$ describing subscriber’s interests, e.g., reducing the costs of calls, SMS, etc.

Concepts from $(\sigma_{\mathbb{R}} \cup \sigma_{\mathbb{I}})$ are used for automation of construction formulas as attributes in formal contexts for association rules mining. Notice that the pair $\langle\sigma_{\mathbb{M}}, \mathcal{A}x\rangle$ forms the ontology of the domain $\mathbb{M}$ [18].

In the next step we introduce a new set of axioms $\mathcal{A}x_S \subseteq F_1(\sigma_{\mathbb{M}})$ and call it the domain axioms. This set will be used for describing various characteristics of payment plans and services provided at present moment of time by a mobile network.

Among other things, these axioms relate personal parameters of a subscriber. The range of parameters contains subscriber traffics denoted by predicates from $\sigma_{\mathbb{P}_1}$ and payments denoted by predicates from $\sigma_{\mathbb{P}_2}$, with regard to activated payment plans from $\sigma_{\mathbb{Q}_1}$ and services from $\sigma_{\mathbb{Q}_2}$.

Axioms $\mathcal{A}x_S$ are true for any case from the domain, and the same statement is true for ontological axioms as well. However, there is a difference between ontological and domain axioms, as the second ones might change over time. Consider the following formula as an example of a domain axiom:

$$(Q_1(x) \rightarrow \neg Q_2(x)), \qquad where \ \ Q_1 \in \sigma_{\mathbb{Q}_1}, Q_2 \in \sigma_{\mathbb{Q}_2}.$$

This formula declares the following: if a subscriber has payment plan Q_1 activated, then service Q_2 cannot be activated for this subscriber. Note that a mobile network company can naturally change its decision for not supporting simultaneously the precise payment plan along with the specific service, at any moment.

3.2 Ontological projections

In order to automate development of the formula set Δ for the sake of finding association rules, we use the Ontological Model of the domain.

Definition 4. An Ontological Model of a domain is a tuple $\langle\mathfrak{A}, T^a, T^s, T^f\rangle$, where T^a is an analytical theory of the domain, T^s is a theory of the domain, and T^f is a fuzzy theory of the model $\mathfrak{A}_{\mathbb{M}}$.

The analytical theory T^a of the domain under consideration is axiomatized by the sentences $\mathcal{A}x_a$ which are axioms of the domain ontology. A theory T^s of the domain is axiomatized by the axioms $\mathcal{A}x_S$ of the domain.

Formula definitions of predicates from $\sigma_{\mathbb{Q}}$ (which present payment plans and services) are defined by construction of ontological projection.

Definition 5. Consider the Ontological Model $\langle \mathfrak{A}, T^a, T^S, T^f \rangle$, let $Q \in \sigma_{\mathbb{Q}}$. Denote

$$S_Q = \{\varphi \in F_1(\sigma_{\mathbb{R}}) | \, T^a \vdash (Q(x) \to \varphi(x))\}.$$

An ontological projection of the predicate Q on the signature $\sigma_{\mathbb{R}}$ is the formula

$$\varphi_Q^{\sigma_{\mathbb{R}}}(x) = \&\{P(x) | \, P \in \sigma_{\mathbb{R}} \; and \; P(x) \in S_Q\}.$$

A projection of the predicate Q on the set of formulas $F_1(\sigma_{\mathbb{R}})$ is the formula

$$\psi_Q^{\sigma_{\mathbb{R}}}(x) = \&S_Q = \&\{\varphi(x) | \, \varphi(x) \in S_Q\}.$$

Let us consider the formal context $C_{\mathfrak{A}}^{\Delta} = (K_{\mathfrak{A}}, \Delta, \vDash)$. We search association rules with the following requirements:

1) a) the premise of the association rule is included in the set $\Delta \upharpoonright \sigma_{\mathbb{P}}$ or
   b) the premise of the association rule is included in the set $\Delta \upharpoonright (\sigma_{\mathbb{P}} \cup \sigma_{\mathbb{Q}})$;
2) a) the conclusion of the association rule belongs to the set $\Delta \upharpoonright \sigma_{\mathbb{Q}}$ or
   b) the conclusion of the association rule belongs to the set $\Delta \upharpoonright \sigma_{\mathbb{R}}$ or
   c) the conclusion of the association rule belongs to the set $\Delta \upharpoonright (\sigma_{\mathbb{R}} \cup \sigma_{\mathbb{I}})$;
3) the support and the confidence of the rules are higher than specified limits.

Notice that the set of association rules of the formal context $C_{\mathfrak{A}}^{\Delta} = (K_{\mathfrak{A}}, \Delta, \vDash)$ is included in the fuzzy theory T^f of the model $\mathfrak{A}_{\mathbb{M}}$ [26, 27].

Then the software system automatically processes obtained association rules. For example, consider an association rule with one-element conclusion P belonging to $\sigma_{\mathbb{R}}$. This rule will be transformed into association rule with the same premise, but the conclusion of the new rule will be one-variable formula from Δ which is a disjunction of all predicates $Q_i \in \sigma_{\mathbb{Q}}$ such that P belongs to the ontological projection of Q_i.

4 Software Implementation

Using the results of the presented investigation, we have developed a software for mining association rules in the formal context $(K_{\mathfrak{A}}, \Delta_{\sigma}^{\vee}, \vDash)$. We have found out that adding predicates from $\sigma_{\mathbb{R}}$ to the formal context gives us the possibility to find association rules with high confidence and support. Conclusions of such rules are transformed into disjunctions of predicates from $\sigma_{\mathbb{Q}}$ with the help of the operator of ontology projection. Obtained association rules seem to be useful for mobile network companies. The software processes the impersonal data for more than 10 million subscribers. This is information for one month of mobile network using by subscribers.

The set of characteristics of subscribers contains more than 90 different items[1]:

1) Personal features of subscriber,
2) Attributes that describe calls made by subscriber,
3) Attributes that describe the mode of using Internet
4) Attributes that describe the mode of using SMS,
5) Attributes that describe the mode of using MMS,
6) Attributes that describe the mode of using LBS (Location Based Services),
7) List of mobile services that were connected to subscriber,
8) Payment plan that is used by subscriber.

The total amount of services that can be connected to subscriber is more than 90. The total count of different payment plans is more than 1200.

Thus, we have more than 10 million objects and nearly 1400 attributes. Part of attributes is quantitative, most part of attributes (more than 1200) are binary.

Let us notice that attributes of connected payment plans, services and personal attributes are always filled. That is why we use only quantitative attributes for density calculation. We calculate data density as follows: $\frac{P}{M*N}$, where P is the number of non-zero subscribers' attributes, M is the total number of subscribers, N is the number of quantitative attributes. For our data the data density is equal to 0.043.

The data is stored in a file with Basket format. Basket is one of standard formats for storing data of "objects-attributes" type in R.

Let us consider an example of association rules which have conclusions consisting of payment plans providing access to the Internet. The predicate $P(x) \in \sigma_{\mathbb{R}}$ denotes that subscriber's payment plan includes unlimited Internet traffic of the special kind[1]. The payment plans having the unlimited Internet traffic of this kind are $Q_1, Q_2, Q_3 \in \sigma_{\mathbb{Q}_1}$, where Q_1 is "Unlimited", Q_2 is "United", and Q_3 is "Online". These payment plans provide unlimited access to Internet with different connection speed and different price. Formally, in terms of ontological projections, it means that $P(x) \in S_{Q_1}$, $P(x) \in S_{Q_2}$, $P(x) \in S_{Q_3}$ and $P(x) \notin S_Q$ for every $Q \in \sigma_{\mathbb{Q}} \backslash \{Q_1, Q_2, Q_3\}$.

Mined association rules have premises with various sets of personal features of subscribers from $\sigma_{\mathbb{P}}$ and the conclusion $P(x)$. The automatically chosen rules have rather high confidence and support (see examples 1 and 2, table 1).

After that the predicate $P(x)$ is substituted by the equivalent disjunction $(Q_1 \vee Q_2 \vee Q_3)$ in the conclusions of the association rules. Table 1 shows that substituting the disjunction $(Q_1 \vee Q_2 \vee Q_3)$ by any of these predicates Q_i notably decreases both confidence and support of the association rules.

Thus, the new association rules generated by the algorithm in the extended formal context $(K_{\mathfrak{A}}, \Delta_{\sigma}^{\vee}, \vDash)$ have higher support and confidence as compared to rules with the same premise which may be found in the original formal context $(K_{\mathfrak{A}}, \Delta_{\sigma}^{\vee}, \vDash)$.

Table 1. Examples of association rules[1].

	Rule	Support	Confidence
Example 1	$\{\boldsymbol{P_1, \dots, P_n}\} \rightarrow \boldsymbol{P}$	**11%**	**91%**
	$\{P_1, \dots, P_n\} \rightarrow Q_1$	6%	50%
	$\{P_1, \dots, P_n\} \rightarrow Q_2$	3%	23%
	$\{P_1, \dots, P_n\} \rightarrow Q_3$	2%	24%
Example 2	$\{\boldsymbol{P''_1, \dots, P''_l}\} \rightarrow \boldsymbol{P}$	**11%**	**89%**
	$\{P''_1, \dots, P''_l\} \rightarrow Q_1$	4%	35%
	$\{P''_1, \dots, P''_l\} \rightarrow Q_2$	5%	38%

[1] Due to NDA, the details of the attribute list and characteristics P_i, Q_i cannot be given. So in the examples below, the real names of characteristics P_i and Q_i have been changed.

	$\{P_1'', \dots, P_l''\} \to Q_3$	2%	18%
Example 3	$\{\boldsymbol{P_1''', \dots, P_l'''}\} \to \boldsymbol{\varphi}$	**10%**	**82%**
	$\{P_1''', \dots, P_l'''\} \to Q_1$	5%	40%
	$\{P_1''', \dots, P_l'''\} \to Q_2$	4%	38%
	$\{P_1''', \dots, P_l'''\} \to Q_3$	0.01%	0.08%
	$\{P_1''', \dots, P_l'''\} \to T_1$	4%	45%
	$\{P_1''', \dots, P_l'''\} \to T_2$	1%	5%
	$\{P_1''', \dots, P_l'''\} \to T_3$	0.01%	0.06%
	$\{P_1''', \dots, P_l'''\} \to T_4$	1%	3%
	$\{P_1''', \dots, P_l'''\} \to (T_1 \vee T_2 \vee T_3 \vee T_4)$	6%	51%

If we would process association rules just in the formal context $(K_{\mathfrak{A}}, \Delta_\sigma^\vee, \vDash)$ without using the signature $\sigma_{\mathbb{R}}$, then many conclusions of mined rules will be non-meaningful disjunctions. Let us consider Example 3 in Table 1. Here $\varphi = (Q_1 \vee Q_2 \vee Q_3 \vee T_1 \vee T_2 \vee T_3 \vee T_4) \in \Delta_\sigma^\vee$, services Q_i provide unlimited Internet, and services T_i provide unlimited SMS. Here T_1 is "unlimited free SMS for month with a fixed price", T_2 is "1000 free SMS for month with a fixed price", T_3 is "unlimited cheap SMS", and T_4 is "discount for SMS, using with special conditions". The confidence of the association rule $\{P_1''', \dots, P_l'''\} \to \varphi$ is high enough. The value is much greater than the confidence of rules $\{P_1''', \dots, P_l'''\} \to Q_i$ and $\{P_1''', \dots, P_l'''\} \to T_i$, but the mobile operator cannot use this association rule for recommendations, because it contains non-related services Q_i and T_i in the conclusion. However, if we consider the association rule $\{P_1''', \dots, P_l'''\} \to (T_1 \vee T_2 \vee T_3 \vee T_4)$, we can see that this rule has low confidence.

5 Conclusion

The paper is devoted to methods for identifying payment plans and services by mobile operators which would be most useful for the given mobile network subscribers. We use the Case Model $\langle K_{\mathfrak{A}}, \sigma \rangle$ for mobile subscriber's behavior description. The Case Model is based on depersonalized subscribers' data provided by mobile operator. Objects (elements of the model) are mobile subscribers. The signature of the Case Model consists of unary predicates. These predicates describe individual subscriber's features (accruals, traffics) or features of payment plans and services. We construct the formal context $(K_{\mathfrak{A}}, \Delta, \vDash)$ based on the Case Model. Then we mine association rules describing payment plans and services that are commonly used by subscribers with given features. After that we consider the formal context $(K_{\mathfrak{A}}, \Delta_\sigma, \vDash)$. Our experiments show that interesting association rules have low confidence values in this context. That is why they cannot be used by mobile operator for any recommendations.

To improve association rules quality we deal with an extension of this formal context, the formal context $(K_{\mathfrak{A}}, \Delta_\sigma^V, \vDash)$. Using this context we can find association rules with high confidence. However, a big part of mined rules have conclusions which are disjunctions of non-related services, e.g. ‘Song instead of Beep’ and ‘Unlimited Internet’. That is why such association rules could not be used for recommendations

Finally, we consider enriched signature $\sigma_{\mathbb{M}}$ instead of the signature σ to find semantically useful disjunctions. Signature $\sigma_{\mathbb{M}}$ contains predicates that describe specific features of payment plans and services. Using the formal context $(K_{\mathfrak{A}}, \Delta_{\sigma_{\mathbb{M}}}, \vDash)$ we compute association rules such that their conclusions are predicates of the signature $\sigma_{\mathbb{R}}$. We transform the obtained association rules into association rules of the formal context $(K_{\mathfrak{A}}, \Delta_{\sigma}^{V}, \vDash)$ using the Ontological Model $\langle K_{\mathfrak{A}}, T^{a}, T^{S}, T^{f} \rangle$. We substitute the predicate in the conclusion of an association rule by disjunction of predicates of the initial signature σ. As the result we obtain association rules of the formal context $(K_{\mathfrak{A}}, \Delta_{\sigma}^{V}, \vDash)$. These rules have high confidence and support, and the conclusions of these rules are completely meaningful for the mobile network operator as well as for mobile network subscribers. Mined association rules allow making recommendations for customers who will be interested in information about these services and tariffs.

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