

A Fresh View on Fuzzy FCA and Mathematical Morphology

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Abstract. In this paper we discuss how biresiduations provide a unifying paradigm for fuzzy formal concept analysis and mathematical morphology. In particular we provide constructions of morphological operators such as dilation and erosion within the framework of convolution algebras of action networks (also known as small covariant categories) over join complete semirings. This unifies and generalizes previous contributions in mathematical morphology, such as the work of Isabelle Bloch. Further we show how decomposition, factorization, and hedges naturally align themselves in the framework of fuzzy formal concept analysis.

Keywords: biresiduation, fuzzy formal concept analysis, hedges, factorization, decomposition, linear algebra, complete monoids

1 Introduction

In our paper we outline the fundamental role of biresiduation in combination with biadditivity for a common view on fuzzy FCA and mathematical morphology. While the concept of biresiduation is an important paradigm of algebraic logic, the concept of biadditivity is rooted in linear algebra (over complete monoids and complete semirings). The close relationship between both approaches is reflected in the bijective correspondence of the residuated complete lattices and the join complete semirings on a fixed set. Our considerations yield a major application for a better understanding of mathematical morphology: We achieve this by constructing and investigating convolution algebras of action networks over join complete semirings. Finally, as a tool for information reduction, we also discuss the role of hedges on residuated complete lattices and their relationship with substructures of join complete semirings, and more generally within the category of biadditive setups.

Important literature for (fuzzy) FCA is given by [1,2,3,4,5], regarding factor analysis, especially by [6], regarding hedges [7]. Belohlávek gives an overview on approaches to fuzzy concept analysis in [8] which is an update of [9].

We assume the reader to be familiar with ordered sets and formal concept analysis as exposed in [5] and [1]. Profound information on residuation theory can be found, for instance, in [10].

Over the last two decades, major contributions to mathematical morphology go back to Isabelle Bloch [11,12,13].
This paper is based on [14].

2 Biresiduation

One of the key concepts for accessing FCA and its generalizations is that of a biresiduated map. We start by recalling the definition of a residuated map.

Definition 1 (residuated map, adjunction). Let $\mathbb{P}_1 := (P_1, \leq)$ and $\mathbb{P}_2 := (P_2, \leq)$ be ordered sets. Then a map $f : P_1 \rightarrow P_2$ is *residuated* from \mathbb{P}_1 to \mathbb{P}_2 if there exists a map $f^+ : P_2 \rightarrow P_1$ such that

$$fp_1 \leq p_2 \iff p_1 \leq f^+(p_2).$$

Here, the map f^+ is uniquely determined by f and is called the residual of f . The pair (f, f^+) is called an *adjunction* w.r.t. $(\mathbb{P}_1, \mathbb{P}_2)$. In case $\mathbb{P}_1 = \mathbb{P}_2$, we will say that (f, f^+) forms an adjunction on \mathbb{P}_1 .

Remark 1. A map between complete lattices is residuated if and only if it is *completely join preserving*, that is, $f(\sup X) = \sup(fX)$ holds for all $X \subseteq P_1$ in the above setting.

The following definition is central for this paper.

Definition 2 (biresiduation). Let $\mathbb{P}_1 := (P_1, \leq)$, $\mathbb{P}_2 := (P_2, \leq)$, and $\mathbb{P} := (P, \leq)$ be ordered sets. Then a map

$$\otimes : P_1 \times P_2 \rightarrow P$$

will be called a *biresiduation* w.r.t. $\mathfrak{P} := (\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$ if there exist maps

$$\overset{\otimes}{\rightarrow} : P_1 \times P \rightarrow P_2$$

$$\overset{\otimes}{\leftarrow} : P \times P_2 \rightarrow P_1$$

such that

$$p_1 \leq (p \overset{\otimes}{\leftarrow} p_2) \iff (p_1 \otimes p_2) \leq p \iff p_2 \leq (p_1 \overset{\otimes}{\rightarrow} p)$$

holds for all $p_1 \in P_1$, $p_2 \in P_2$, and $p \in P$.

In case $\mathbb{P}_1 = \mathbb{P}_2 = \mathbb{P}$, we say that \otimes forms a biresiduation w.r.t. \mathbb{P} . If in addition $(P, \otimes, \varepsilon)$ is a monoid then $(\mathbb{P}, \otimes, \varepsilon)$ will be referred to as *residuated ordered set*. If in particular \mathbb{P} forms a complete lattice, $(\mathbb{P}, \otimes, \varepsilon)$ is a *residuated complete lattice*.

Remark 2. A biresiduation is characterized as a map \otimes which is residuated in both arguments. In particular a biresiduation is isotone in each argument.

Example 1. Any two sets G and M give rise to a biresiduation as follows: The map

$$\otimes : 2^G \times 2^M \rightarrow 2^{G \times M}, (A, B) \mapsto A \times B$$

is a biresiduation w.r.t. $(2^G, 2^M, 2^{G \times M})$, where 2^G denotes the power set lattice of G . This follows immediately since power set lattices are complete with the supremum operation as set union. In particular for $I \subseteq G \times M$ we consider the context (G, M, I) : For all $A \subseteq G$ and all $B \subseteq M$ we have

$$(A \times B) \subseteq \alpha \iff A \subseteq (B \overset{\otimes}{\rightarrow} \alpha)$$

where

$$(B \overset{\otimes}{\rightarrow} \alpha) = \bigcup \{H \subseteq G \mid H \times B \subseteq \alpha\} = B'.$$

Dually, we get

$$(\alpha \overset{\otimes}{\leftarrow} A) = \bigcup \{N \subseteq M \mid A \times N \subseteq \alpha\} = A'.$$

We have recaptured the derivation operators of classical formal concept analysis as residuals of a specific biresiduation, the cartesian product.

Now, it is worth noting the connection between cartesian product and dyadic product as used in linear algebra. We recall the definition of a dyadic product. Given two n -dimensional vectors \mathbf{u}, \mathbf{v} over a semiring S we can define the dyadic product as

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u} \cdot \mathbf{v}^T$$

where the second multiplication is simply matrix multiplication. If S is the well-known boolean 2-element semiring, the dyadic product resembles exactly the cartesian product. So, in a sense, the dyadic product generalizes the cartesian product. From our point of view, this is key to understanding fuzzy concept analysis in terms of biresiduations.

Example 2 (t-Norm). A binary operation \otimes on the real unit interval $[0, 1]$, which is isotone in both arguments, is called *t-Norm* if for all $a, b, c \in [0, 1]$ the following hold:

- (1) $1 \otimes a = a$
- (2) $a \otimes b = b \otimes a$
- (3) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$

The t-norm \otimes is *left continuous* if for all $a \in [0, 1]$ and $\alpha \in [0, 1]^I$ (where I is an arbitrary index set) the following holds:

$$a \otimes \left(\sup_{i \in I} \alpha_i \right) = \sup_{i \in I} (a \otimes \alpha_i)$$

From the terminology introduced above, a map $\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a left continuous t-norm if and only if $([0, 1], \leq, \otimes, 1)$ forms a residuated complete lattice. Among the various left continuous t-norms we point out the Gödel t-norm

$$\begin{aligned} \otimes : [0, 1] \times [0, 1] &\rightarrow [0, 1], \\ (a, b) &\mapsto \min(a, b) \end{aligned}$$

and the Łukasiewicz t-norm

$$\begin{aligned} \otimes : [0, 1] \times [0, 1] &\rightarrow [0, 1], \\ (a, b) &\mapsto \max(a + b - 1, 0) \end{aligned}$$

These t-norms will be used in our visualizations for mathematical morphology in figure 1 and 2.

In the next section, we will highlight that residuals of a biresiduation play a crucial role in FCA and its abstractions.

3 Abstract Concepts and Maximal Rectangles

Let $\mathbb{P}_1 := (P_1, \leq)$, $\mathbb{P}_2 := (P_2, \leq)$ and $\mathbb{P} = (P, \leq)$ be ordered sets and \otimes a biresiduation w.r.t. $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$. We define

$$\boxed{f}_{\otimes} := P_1 \times P_2$$

to be its set of *formal rectangles*. If $\alpha \in P$ then

$$\mathcal{K} := (\otimes, \alpha)$$

is called an *abstract context* w.r.t. $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$. Given such an abstract context we can define the set of formal rectangles w.r.t. \mathcal{K} as

$$\boxed{f}_{\mathcal{K}} := \{(u, w) \in \boxed{f}_{\otimes} \mid u \otimes w \leq \alpha\}.$$

On a formal rectangle, we can apply our biresiduation operation to yield an (*actual*) *rectangle*. We define

$$\square_{\mathcal{K}} := \{u \otimes w \mid (u, w) \in \boxed{f}_{\mathcal{K}}\}$$

to be the set of (*actual*) *rectangles* w.r.t. \mathcal{K} and

$$\boxed{mf}_{\mathcal{K}} := \max \boxed{f}_{\mathcal{K}}$$

to be the set of maximal rectangles regarding the product order on $P_1 \times P_2$. We define the set of abstract concepts as

$$\mathfrak{BK} := \{(u, w) \in P_1 \times P_2 \mid u \overset{\otimes}{\rightarrow} \alpha = w \ \& \ \alpha \overset{\otimes}{\leftarrow} w = u\}.$$

We order the set of maximal rectangles of an abstract context. For $b = (u_b, w_b), c = (u_c, w_c) \in \boxed{\text{mf}}_{\mathcal{K}}$ we set

$$b \leq_{\mathcal{K}} c : \iff u_b \leq_{P_1} u_c \iff w_c \leq_{P_2} w_b.$$

Now we can define an *abstract concept order* as

$$\underline{\mathfrak{B}}\mathcal{K} := (\mathfrak{B}\mathcal{K}, \leq_{\mathcal{K}}).$$

In case $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$ is a triple of complete lattices, $\underline{\mathfrak{B}}\mathcal{K}$ forms a complete lattice, called the abstract concept lattice of \mathcal{K} .

As usual in FCA, let us abbreviate $u \overset{\otimes}{\rightarrow} \alpha$ as u' , the derivation of u w.r.t. $\mathcal{K} := (\otimes, \alpha)$, and similarly, $\alpha \overset{\otimes}{\leftarrow} w$ as w' . We show that even in our rather abstract setting we can talk about maximal rectangles being the abstract concepts.

Proposition 1. *Let $\mathcal{K} := (\otimes, \alpha)$ be an abstract context w.r.t. $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$ and define $\gamma : P_1 \rightarrow P, x \mapsto (x'', x')$ and $\mu : P_2 \rightarrow P, x \mapsto (x', x'')$. Then*

1. $\mathfrak{B}\mathcal{K} = \text{im}(\gamma) = \text{im}(\mu)$
2. $\boxed{\text{mf}}_{\mathcal{K}} = \mathfrak{B}\mathcal{K}$

We call $(p_1, p_2) \in \boxed{\text{f}}_{\mathcal{K}}$ a *decomposition* of \mathcal{K} if $p_1 \otimes p_2 = \alpha$. If additionally $(p_1, p_2) \in \boxed{\text{mf}}_{\mathcal{K}}$ we call (p_1, p_2) a *conceptual decomposition* of \mathcal{K} .

Corollary 1. *Let $\mathcal{K} = (\otimes, \alpha)$ be an abstract context w.r.t. $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$. If (p_1, p_2) is a decomposition of \mathcal{K} there exists a conceptual decomposition (q_1, q_2) of \mathcal{K} with $p_1 \leq q_1$ and $p_2 \leq q_2$.*

Proof. For instance, set $(q_1, q_2) := \gamma(p_1)$. □

4 Biadditivity

We complement the approach sketched above (where ordered sets are used as basic structures) by using complete monoids [15] as basic structures.

Definition 3 (complete monoid). A quadruple $\mathcal{A} := (A, +, 0, \Sigma)$ is called a complete monoid if $(A, +, 0)$ is a commutative monoid and Σ assigns to every $\alpha \in A^I$ (for an arbitrary index set I) an element $\Sigma\alpha =: \Sigma_{i \in I} \alpha i$ of A such that

- (1) $\Sigma\alpha = 0$ if $\alpha i = 0$ for all $i \in I$
- (2) $\Sigma\alpha = \alpha i$ if $I = \{i\}$
- (3) $\Sigma\alpha = \alpha i + \alpha j$ if $I = \{i, j\}$ and $i \neq j$
- (4) $\Sigma\alpha = \Sigma\beta$ for every partition \mathcal{T} of I and β given by $\mathcal{T} \rightarrow A, T \mapsto \Sigma\alpha|_T$

Definition 4 (join complete monoid). A *join complete monoid* is a complete monoid $\mathcal{A} := (A, +, 0, \Sigma)$ such that for every non-empty set I and every $a \in A$ it follows $\sum_{i \in I} a = a$.

Remark 3. For every complete lattice $\mathbb{L} := (L, \leq)$ let $\mathcal{A}(\mathbb{L}) := (L, +, 0, \Sigma)$ be the join complete monoid, where $x + y := \sup_{\mathbb{L}}\{x, y\}$ for all $x, y \in L$ and $0 := \sup_{\mathbb{L}}\emptyset$ and $\Sigma\alpha := \sup_{\mathbb{L}}\{\alpha(i) \mid i \in I\}$ for every index set I and all $\alpha \in L^I$. If on the other hand, $\mathcal{A} := (A, +, 0, \Sigma)$ is a join complete monoid then $\mathbb{L}(\mathcal{A}) := (A, \leq)$ is a complete lattice, provided $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in A$. This observation establishes for every set A a bijective correspondence between all complete lattices on A and all join complete monoids on A .

$\mathfrak{L} := (\mathbb{L}_1, \mathbb{L}_2, \mathbb{L})$ is a triple of complete lattices then $\mathcal{A}(\mathfrak{L}) := (\mathcal{A}(\mathbb{L}_1), \mathcal{A}(\mathbb{L}_2), \mathcal{A}(\mathbb{L}))$. We define the analogue of biresiduations for complete monoids.

Definition 5 (biadditive map). Let $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$ be a triple of complete monoids. Then a biadditive map w.r.t. \mathfrak{A} is defined as a map $\otimes : A_1 \times A_2 \rightarrow A$ such that

$$\Sigma\beta \otimes \Sigma\gamma = \Sigma_{(i,j) \in I \times J} \beta i \otimes \gamma j$$

holds for all $\beta \in A_1^I$ and $\gamma \in A_2^J$ for arbitrary index sets I, J .

The next definition will also be useful in the section concerning hedges.

Definition 6 (biadditive setup, morphism). Let $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$ be a triple of complete monoids and let \otimes be a biadditive map on \mathfrak{A} . Then we call the pair (\mathfrak{A}, \otimes) a *biadditive setup*. In case $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, we say that \otimes is a *biadditive operation* on \mathcal{A} , and (\mathcal{A}, \otimes) forms a *biadditive setup*.

Given two biadditive setups (\mathfrak{A}, \otimes) and (\mathfrak{U}, \otimes) we call $\Phi := (\varphi_1, \varphi_2, \varphi)$ a morphism from (\mathfrak{U}, \otimes) to (\mathfrak{A}, \otimes) if $\varphi_1, \varphi_2, \varphi$ are morphisms from U_1, U_2 , and U to A_1, A_2 , and A , respectively, and for all $u_1 \in U_1$ and $u_2 \in U_2$ we have

$$\varphi_1(u_1) \otimes \varphi_2(u_2) = \varphi(u_1 \otimes u_2).$$

Biadditive setups and their morphisms obviously form a category. In particular, (\mathfrak{U}, \otimes) is a *substructure* of (\mathfrak{A}, \otimes) if U_1, U_2 , and U are complete submonoids of A_1, A_2 , and A , respectively, and $U_1 \otimes U_2 \subseteq U$. Clearly, The image of a morphism induces a substructure.

Definition 7 (complete semiring). A tuple $\mathfrak{R} := (R, +, \otimes, 0, 1, \Sigma)$ is a complete semiring if the following properties are satisfied:

- (1) $\mathfrak{R}_{add} := (R, +, 0, \Sigma)$ is a complete monoid,
- (2) $\mathfrak{R}_{mult} := (R, \otimes, 1)$ is a monoid with $1 \neq 0$,
- (3) the following distributive laws hold for all $a \in R$, $\alpha \in R^I$:

$$\begin{aligned} a \otimes \left(\sum_{i \in I} \alpha i \right) &= \sum_{i \in I} (a \otimes \alpha i) \\ \left(\sum_{i \in I} \alpha i \right) \otimes a &= \sum_{i \in I} (\alpha i \otimes a). \end{aligned}$$

Remark 4. If $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ is a complete semiring then $(\mathfrak{R}_{add}, \otimes)$ forms a biadditive setup.

Definition 8. A *join complete semiring* is a complete semiring \mathfrak{R} such that \mathfrak{R}_{add} is a join complete monoid.

Remark 5. If $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ is a given join complete semiring then $\mathcal{L}(\mathfrak{R}) := (\mathbb{L}(\mathfrak{R}_{add}), \otimes, 1)$ forms a complete residuated lattice. If on the other hand $\mathcal{L} := (L, \otimes, \varepsilon)$ with $\mathbb{L} = (L, \leq)$ is a given residuated complete lattice then $\mathfrak{R}(\mathcal{L}) := (L, +, \otimes, 0, \varepsilon, \sum)$ with $(L, +, 0, \sum) := \mathcal{A}(\mathbb{L})$ is a join complete semiring. This establishes a bijection between all join complete semirings on a set R and all complete residuated lattices on R . In particular for every join complete semiring $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ there exist maps $\overset{\otimes}{\rightarrow}$ and $\overset{\otimes}{\leftarrow}$ from $R \times R$ to R such that for all $r, s, t \in R$ the following holds:

$$r \leq (t \overset{\otimes}{\leftarrow} s) \iff (r \otimes s) \leq t \iff s \leq (r \overset{\otimes}{\rightarrow} t)$$

A more extensive discussion of the interplay between biresiduation and biadditivity will be presented in section 6.

Proposition 2. Let (\mathfrak{A}, \otimes) be a biadditive setup where $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$. For sets G and M define $\boxtimes : A_1^G \times A_2^M \rightarrow A^{G \times M}$ where

$$(u \boxtimes w)(g, m) := u(g) \otimes w(m).$$

Then \boxtimes forms a biadditive map w.r.t. $(A_1^G, A_2^M, A^{G \times M})$ which is also known as the dyadic product.

More generally, we have the following construction.

Proposition 3. Let (\mathfrak{A}, \otimes) be a biadditive setup where $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$. For sets G , H , and M define

$$* : A_1^{G \times H} \times A_2^{H \times M} \rightarrow A^{G \times M}$$

where

$$(\beta * \eta)(g, m) := \sum_{h \in H} \beta(g, h) \otimes \eta(h, m).$$

Then $*$ forms a biadditive map w.r.t. $(A_1^{G \times H}, A_2^{H \times M}, A^{G \times M})$ – which, as a matter of fact, is the matrix product.

If (\mathfrak{A}, \otimes) is a biadditive setup where $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$ and $\alpha \in A$ then a family $(u_h, w_h)_{h \in H} \in (A_1 \times A_2)^H$ will be called a sum-decomposition of (\otimes, α) if

$$\alpha = \sum_{h \in H} u_h \otimes w_h.$$

An important observation is the following

Proposition 4. Let (\mathfrak{A}, \otimes) be a biadditive setup where $\mathfrak{A} := (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A})$. Then for sets G , H , M and $\alpha \in A^{G \times M}$ the following holds

1. If (β, η) is a decomposition of $(*, \alpha)$, that is, $\beta \in A_1^{G \times H}$ and $\eta \in A_2^{H \times M}$ such that $\alpha = \beta * \eta$, then $(u_h, w_h)_{h \in H}$ is a sum-decomposition of (\boxtimes, α) where $u_h := \beta(\cdot, h)$ and $w_h := \eta(h, \cdot)$.
2. Conversely, if $(u_h, w_h)_{h \in H}$ is a sum-decomposition of (\boxtimes, α) then (β, η) is a decomposition of $(*, \alpha)$ where $\beta : G \times H \rightarrow A_1, (g, h) \mapsto u_h g$ and $\eta : H \times M \rightarrow A_2, (h, m) \mapsto w_h m$.

Remark 6. An important situation where this proposition can be applied occurs when $\mathfrak{R} = (R, +, \otimes, 0, 1, \Sigma)$ is a complete semiring, since then $(\mathfrak{R}_{add}, \otimes)$ is a biadditive setup as already mentioned in remark 4.

In mathematical morphology the concept of dilation is fundamental, and crucially involves *convolution* in a general setting which will be layed out in the following.

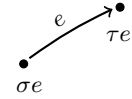
5 Construction of Convolution Algebras

The ingredients of our modelling approach for the construction of convolution algebras are action networks and complete semirings. For this we still need to introduce networks and action networks.

Definition 9 (network). A triple $\mathcal{G} := (V, E, \varrho)$ will be called a *network* (*directed multigraph*) if V and E are sets and $\varrho : E \rightarrow V \times V$ is a map. In this context we interpret V as the *set of nodes*, E as the *set of edges*, and ϱ as the *structure map* of \mathcal{G} .

Additional Remark: The structure map ϱ of \mathcal{G} induces two maps

$$\begin{aligned} \sigma : E &\rightarrow V, e \mapsto \sigma e \quad \text{and} \\ \tau : E &\rightarrow V, e \mapsto \tau e \end{aligned}$$



satisfying $\varrho e = (\sigma e, \tau e)$ for all $e \in E$; here we consider σe as *source node* and τe as *target node* of e .

A pair of edges (c, d) will be called *sequential* if the target node of c is equal to the source node of d . The set of all sequential pairs of edges will be denoted by $E^{(2)}$. Similarly $E^{(3)}$ will denote the set of all triples of edges (c, d, e) such that (c, d) and (d, e) are sequential pairs.

Next we consider networks with additional structure. In this situation, edges will be interpreted as actions which allow concatenation of sequential pairs of actions.

Definition 10 (action network). A triple $\mathbb{G} := (\mathcal{G}, *, id)$ will be called an *action network* (*small covariant category*) if $\mathcal{G} := (V, E, \varrho)$ is a network and

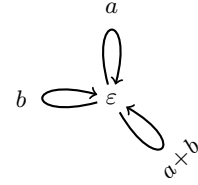
$$* : E^{(2)} \rightarrow E, (c, d) \mapsto c * d \quad \text{and} \quad id : V \rightarrow E$$

are maps satisfying:

$$\begin{aligned}\varrho(c * d) &= (\sigma c, \tau d) \text{ for all } (c, d) \in E^{(2)} \\ (c * d) * e &= c * (d * e) \text{ for all } (c, d, e) \in E^{(3)} \\ \varrho(idp) &= (p, p) \text{ for all } p \in V \\ id(\sigma c) * c &= c = c * id(\tau c) \text{ for all } c \in E.\end{aligned}$$

We interpret E as set of *actions* and $*$ as *concatenation map*, which maps every sequential pair of actions (c, d) to its concatenation $c * d$. Furthermore, idp denotes the *passive action* at node p .

Example 3. Any monoid $\mathbb{M} := (M, *, \varepsilon)$ can be interpreted as action network with $\{\varepsilon\}$ as the singleton node set, M as the set of edges, and $*$ as concatenation map as well as $id : \{\varepsilon\} \rightarrow M, \varepsilon \mapsto \varepsilon$.



Example 4. Any preordered set $\mathbb{P} := (P, R)$ can be interpreted as action network which has (P, R, ρ) as underlying network, where $\rho : R \rightarrow V \times V, (p, q) \mapsto (p, q)$, and $* : R^{(2)} \rightarrow R, ((p, t), (t, q)) \mapsto (p, q)$ as concatenation map and furthermore $id : P \rightarrow R, p \mapsto (p, p)$ as passivity map.

Construction 1 (convolution algebra). Let $\mathfrak{R} := (R, +, \cdot, 0, 1, \sum)$ be a complete semiring and let $\mathbb{G} := (\mathcal{G}, *, id)$ be an action network with underlying network $\mathcal{G} := (V, E, \varrho)$; then the *convolution algebra* of \mathbb{G} over \mathfrak{R} is given by

$$\mathfrak{R}[\mathbb{G}] := (R^E, +, *, \mathbf{O}, \mathbf{I}, \sum),$$

where for every index set I for all $i \in I$ and $u_i, u, w \in R^E$ as well as $e \in E$ the following hold:

$$\begin{aligned}(u + w)e &:= ue + we \\ \left(\sum_{i \in I} u_i\right)e &:= \sum_{i \in I} (u_i e) \\ (u * w)e &:= \sum_{(c, d) \in Split_{\mathbb{G}}(e)} uc \cdot wd\end{aligned}$$

with $Split_{\mathbb{G}}(e) := \{(c, d) \in E^{(2)} \mid c * d = e\}$ and

$$\begin{aligned}\mathbf{O} : E &\rightarrow R, e \mapsto 0 \\ \mathbf{I} : E &\rightarrow R, e \mapsto \begin{cases} 1 & \text{for } e \in idV \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Remark 7. If u and w are as above then the product $u * w$ will be called the *convolution* of u with w .

Remark 8. In case $\mathbb{M} := (M, *, \varepsilon)$ is a monoid then $\mathfrak{A}[\mathbb{M}]$ is defined as $\mathfrak{A}[\mathbb{G}]$ where \mathbb{G} is the action network associated with \mathbb{M} and $\mathfrak{A}[\mathbb{M}]$.

6 Combining Biresiduation and Biadditivity

The following fact will help us to combine biresiduation and biadditivity, which will be relevant for fuzzy FCA and also for mathematical morphology:

Proposition 5. Let $\mathfrak{L} := (\mathbb{L}_1, \mathbb{L}_2, \mathbb{L})$ be a triple of complete lattices and let $\otimes : L_1 \times L_2 \rightarrow L$ be a map. Then \otimes is biresiduated w.r.t. \mathfrak{L} if and only if \otimes is biadditive w.r.t. $\mathcal{A}(\mathfrak{L})$.

Theorem 1. Let $\mathfrak{L} := (\mathbb{L}_1, \mathbb{L}_2, \mathbb{L})$ be a triple of complete lattices and let \otimes be a biresiduation w.r.t. \mathfrak{L} . Then for sets G , R , and M the following holds:

1. \boxtimes is a biresiduation w.r.t. $(L_1^G, L_2^M, L^{G \times M})$.
Hence, for all $u \in L_1^G$ and $w \in L_2^M$ and $\alpha \in L^{G \times M}$ we have

$$u \leq (\alpha \stackrel{\boxtimes}{\leftarrow} w) \iff (u \boxtimes w) \leq \alpha \iff w \leq (u \stackrel{\boxtimes}{\rightarrow} \alpha).$$

Also,

$$(\alpha \stackrel{\boxtimes}{\leftarrow} w)(g) = \inf_{L_1} \{ \alpha(g, m) \stackrel{\otimes}{\leftarrow} w(m) \mid m \in M \}$$

for all $g \in G$ and

$$(u \stackrel{\boxtimes}{\rightarrow} \alpha)(m) = \inf_{L_2} \{ u(g) \stackrel{\otimes}{\rightarrow} \alpha(g, m) \mid g \in G \}$$

for all $m \in M$.

2. $*$ is a biresiduation w.r.t. $(L_1^{G \times H}, L_2^{H \times M}, L^{G \times M})$.
Hence, for all $\beta \in L_1^{G \times H}$ and $\eta \in L_2^{H \times M}$ and $\alpha \in L^{G \times M}$ we have

$$\beta \leq \alpha \stackrel{*}{\leftarrow} \eta \iff \beta * \eta \leq \alpha \iff \eta \leq \beta \stackrel{*}{\rightarrow} \alpha.$$

Also,

$$(\alpha \stackrel{*}{\leftarrow} \eta)(g, h) = \inf_{L_1} \{ \alpha(g, m) \stackrel{\otimes}{\leftarrow} \eta(h, m) \mid m \in M \}$$

for all $(g, h) \in G \times H$ and

$$(\beta \stackrel{*}{\rightarrow} \alpha)(h, m) = \inf_{L_2} \{ \beta(g, h) \stackrel{\otimes}{\rightarrow} \alpha(g, m) \mid g \in G \}$$

for all $(h, m) \in H \times M$.

3. Let $\alpha \in L^{G \times M}$ and let (β_0, η_0) be a decomposition of $\mathcal{K} = (*, \alpha)$. Then there exists a conceptual decomposition (β, η) of \mathcal{K} with $\beta_0 \leq \beta$ and $\eta_0 \leq \eta$. The corresponding sum-decomposition $(\beta(\cdot, h), \eta(h, \cdot))_{h \in H}$ of $\mathcal{K}_0 := (\boxtimes, \alpha)$ consists of abstract concepts of \mathcal{K}_0 . Such a sum-decomposition is called *conceptual sum-decomposition*.

Conversely, if $(x_h, y_h)_{h \in H}$ is a sum-decomposition of \mathcal{K}_0 then there exists a conceptual sum-decomposition $(u_h, w_h)_{h \in H}$ of \mathcal{K}_0 with $x_h \leq u_h$ and $y_h \leq w_h$. The corresponding decomposition (β, η) of \mathcal{K} defined via

$$\beta : G \times H \rightarrow L_1, (g, h) \mapsto u_h(g)$$

and

$$\eta : H \times M \rightarrow L_2, (h, m) \mapsto w_h(m)$$

is a conceptual decomposition of \mathcal{K} . If (β, η) is a conceptual decomposition of \mathcal{K} then, by definition, $\beta = \alpha \leftarrow \eta$ and $\eta = \beta \rightarrow \alpha$.

Proof. Part 1 follows from Propositions 5 and 2. Part 2 follows from Propositions 5 and 3. Part 3 follows from Proposition 5 and 4 together with Corollary. \square

The above theorem extends Theorem 6 from [16]. Note that Part 3 of the above theorem employs a well-known fact from linear algebra: matrix multiplication can be rewritten as summing over the dyadic products of the respective column and row vectors.

Remark 9. Referring to remark 6, the last theorem is connected with fuzzy formal concept analysis in the following way: Let $\mathfrak{R} := (R, +, \otimes, 0, 1, \Sigma)$ be a join complete semiring. Then for sets G, M and $\alpha \in R^{G \times M}$, the triple (G, M, α) will be regarded as *fuzzy context* over \mathfrak{R} having the same concept lattice as the abstract context (\boxtimes, α) w.r.t. $(\mathbb{L}^G, \mathbb{L}^M, \mathbb{L}^{G \times M})$ for $\mathbb{L} := \mathbb{L}(\mathfrak{R}_{add})$. Also the decomposition discussed in the third part of the above theorem applies to this situation.

If we restrict ourselves to the situation of M being a singleton, Theorem 1.1 yields for all $r \in R$ and $u, v \in R^G$

$$u \otimes r \leq v \iff r \leq (u \overset{\otimes}{\rightarrow} v),$$

that is, $u \rightarrow v$ can be interpreted as the degree of u being a subset of v .

Theorem 2. For every complete semiring $\mathfrak{R} := (R, +, \otimes, 0, 1, \Sigma)$ and every action network $\mathbb{G} := (\mathcal{G}, *, id)$ with $\mathcal{G} := (V, E, \varrho)$ the convolution algebra $\mathfrak{R}[\mathbb{G}]$ is a complete semiring.

In case \mathfrak{R} is join complete then so is $\mathfrak{R}[\mathbb{G}]$; consequently $(\mathbb{L}(R^E, +, \mathbf{0}, \Sigma), *, \mathbf{1})$ forms a residuated complete lattice, and for all $u, w \in R^E$ and all $d \in E$ it follows

$$(u \overset{*}{\rightarrow} w)d = \inf\{uc \overset{\otimes}{\rightarrow} w(c * d) \mid c \in E : (c, d) \in E^{(2)}\}.$$

Proof. Straightforward calculation yields that $\mathfrak{R}[\mathbb{G}]$ is a complete semiring, which is join complete if \mathfrak{R} is join complete.

In the latter, it remains to show that for all $u, v, w \in R^E$ the following holds:

$$\begin{aligned}
& \forall d \in E : vd \leq (u \overset{*}{\rightarrow} w)d \\
& \Leftrightarrow v \leq (u \overset{*}{\rightarrow} w) \\
& \Leftrightarrow (u * v) \leq w \\
& \Leftrightarrow \forall e \in E : \sum_{(c,d) \in \text{Split}(e)} uc \otimes vd \leq we \\
& \Leftrightarrow \forall e \in E, \forall (c,d) \in \text{Split}(e) : uc \otimes vd \leq we \\
& \Leftrightarrow \forall (c,d) \in E^{(2)} : uc \otimes vd \leq w(c * d) \\
& \Leftrightarrow \forall (c,d) \in E^{(2)} : vd \leq uc \overset{\otimes}{\rightarrow} w(c * d) \\
& \Leftrightarrow \forall d \in E : vd \leq \inf\{uc \overset{\otimes}{\rightarrow} w(c * d) \mid c \in E : (c,d) \in E^{(2)}\}
\end{aligned}$$

Indeed, the above equivalences immediately imply

$$(u \overset{\otimes}{\rightarrow} w)d = \inf\{uc \overset{\otimes}{\rightarrow} w(c * d) \mid c \in E : (c,d) \in E^{(2)}\} \text{ for all } d \in E.$$

□

Construction 2 (dilation and erosion). Let $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ be a join complete semiring and $\mathbb{G} := (\mathcal{G}, *, id)$ be an action network with $\mathcal{G} := (V, E, \varrho)$. Then an element $\nu \in R^E$ will be considered as *structuring element* of \mathcal{G} over \mathfrak{R} , and the map

$$\delta_\nu : R^E \rightarrow R^E, \mu \mapsto (\nu * \mu)$$

is called the *dilation* via the structuring element ν and the map

$$\varepsilon_\nu : R^E \rightarrow R^E, \mu \mapsto (\nu \overset{*}{\rightarrow} \mu)$$

is the *erosion* via the structuring element ν . In this setting, $\mu \in R^E$ often plays the role of an image, the dilation of which via the structuring element ν is given by

$$(\delta_\nu)\mu = (\nu * \mu).$$

Similarly, the erosion via ν of the image μ is given by

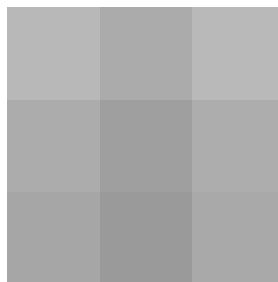
$$(\varepsilon_\nu)\mu = (\nu \overset{*}{\rightarrow} \mu).$$

Remark 10. Here, the pair $(\delta_\nu, \varepsilon_\nu)$ is an adjunction on $\mathbb{L}(R^E, +, \mathbf{0}, \sum)$.

A significant application of mathematical morphology is photo editing. Here we visualize dilation and erosion by using convolution algebras induced via the t-norms introduced previously.



original



Structuring element



dilation



erosion

Fig. 1: Łukasiewicz t-norm



dilation



erosion

Fig. 2: Gödel t-norm

Remark 11 (fuzzy FCA). Referring to construction 2, let $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ be a given join complete semiring and $\mathbb{G} := (\mathcal{G}, *, id)$ be an action network with $\mathcal{G} := (V, E, \varrho)$. In this situation, an element $\eta \in R^E$ will be considered as input image and an element $\nu \in R^E$ will be regarded as structuring element. Then $\mu := (\nu \xrightarrow{*} \eta)$ is the erosion of η via ν . From the viewpoint of fuzzy FCA, μ is the derivation of ν in the abstract context $\mathcal{K} := (*, \eta)$ w.r.t. $(\mathbb{L}^E, \mathbb{L}^E, \mathbb{L}^E)$ for $\mathbb{L} := \mathbb{L}(\mathfrak{R}_{add})$.

7 Integration of Isabelle Bloch's Approach

Isabelle Bloch has been one of the key scientists in developing mathematical morphology and its fuzzifications during the past 20 years (for example we mention [11,17,18,19] and [20,12,13,21]). In her recent papers on the topic, she considered as underlying structure a so-called *space*, which is given by a commutative group $\mathcal{S} := (S, +, O)$.

Construction 3 (dilation and erosion after Bloch - cf. [11]).

Let $\mathcal{S} := (S, +, O)$ be a commutative group and $\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$ a left continuous t-norm. Then Isabelle Bloch defines the *dilation* via a structuring element $\nu : S \rightarrow [0, 1]$ as

$$\begin{aligned} \delta_\nu : [0, 1]^S &\rightarrow [0, 1]^S, \\ \mu &\mapsto \delta_\nu \mu \end{aligned}$$

where

$$(\delta_\nu \mu)x := \sup\{\nu(x - y) \otimes \mu y \mid y \in S\} \text{ for all } x \in S.$$

Similarly, the *erosion* via ν is defined as the map

$$\begin{aligned} \varepsilon_\nu : [0, 1]^S &\rightarrow [0, 1]^S, \\ \mu &\mapsto \varepsilon_\nu \mu \end{aligned}$$

where

$$(\varepsilon_\nu \mu)x := \inf\{\nu(y - x) \overset{\otimes}{\rightarrow} \mu y \mid y \in S\} \text{ for all } x \in S.$$

Here $\delta_\nu \mu$ is the dilation of the image μ via the structuring element ν , and $\varepsilon_\nu \mu$ is the erosion of the image μ via the structuring element ν . Indeed, Bloch proves that the pair $(\delta_\nu, \varepsilon_\nu)$ forms an adjunction on $([0, 1], \leq)$.

Remark 12. By theorem 2 it follows immediately that $(\delta_\nu, \varepsilon_\nu)$, as introduced by Bloch, forms an adjunction on $([0, 1], \leq)$: Indeed, theorem 2 implies for the complete semiring $\mathfrak{R} := ([0, 1], \vee, \otimes, 0, 1, \sup)$ that the pair $(\delta_\nu, \varepsilon_\nu)$ forms an adjunction on $\mathfrak{R}[\mathcal{S}]$, that is, on $\mathbb{L}(\mathfrak{R}[\mathcal{S}])$.

In our final section we will discuss hedges and their role for information reduction.

8 Hedges

Hedges were introduced as parameters controlling the size of a fuzzy concept lattice [7]. In [22], fuzzy concept lattices with hedges are shown to be essentially generalized concept lattices in Krajci's sense. Also, in our framework, they have a natural place. We find that they are intimately connected with the substructures of join-complete semirings. A hedge $*$: $L \rightarrow L$ is defined as a kernel operator on \mathbb{L} (i.e. $a^* \leq b \iff a^* \leq b^*$) such that it preserves the 1 ($1^* = 1$) and weakly preserves implications ($(a \otimes b)^* \leq a^* \otimes b^*$). Note that this definition of a hedge obviously generalizes the notion introduced in [7] (monotonicity for Belohlavek's notion is shown in [22], Lemma 1). The next theorem shows that the theory of hedges can also be formulated in a linear algebraic language within the framework of join-complete semirings.

Theorem 3. Let $\mathcal{L} = (\mathbb{L}, \otimes, \varepsilon)$ be a residuated complete lattice. Then its hedges are in one-to-one correspondence with the substructures of the join-complete semiring $\mathfrak{R}(\mathcal{L})$.

Proof. Since hedges are kernel operators they are in one-to-one correspondence with their kernel systems (a hedge $*$ is mapped to its image set L^*).

Let $*$ be a hedge on \mathcal{L} . We will show that its image set L^* forms a substructure (join-complete sub-semiring) of $\mathfrak{R}(\mathcal{L})$. We know already that L^* is closed under joins, that $1^* = 1 \in L^*$, and that $0^* = 0 \in L^*$. To show that L^* is closed under \otimes we will verify that $a \otimes b = (a \otimes b)^*$ holds for all $a, b \in L^*$; it suffices to prove $a \otimes b \leq (a \otimes b)^*$:

$$\begin{aligned} & a \otimes b \leq a \otimes b \\ \iff & b \leq (a \otimes (a \otimes b))^* \\ \implies & b^* \leq (a \otimes (a \otimes b))^* \\ \implies & b^* \leq (a^* \otimes (a \otimes b))^* \\ \iff & a^* \otimes b^* \leq (a \otimes b)^* \end{aligned}$$

Since $a^* = a$ and $b^* = b$ we conclude the argument.

Let \mathcal{K} be a substructure of a given join-complete semiring $\mathfrak{R}(\mathcal{L})$. Since \mathcal{K} is closed under arbitrary joins it forms a kernel system in \mathbb{L} inducing its kernel operator $*$. Thus, $\mathcal{K} = L^*$ and $1 \in L^*$. It remains to be shown that implications are weakly preserved by $*$:

$$\begin{aligned} & (a \otimes b)^* \leq (a \otimes b) \\ \iff & a \otimes (a \otimes b)^* \leq b \\ \implies & a^* \otimes (a \otimes b)^* \leq b \end{aligned}$$

Since \mathcal{K} is a substructure of $\mathfrak{R}(\mathcal{L})$ the element $a^* \otimes (a \otimes b)^*$ is a kernel and we get $a^* \otimes (a \otimes b)^* \leq b^*$, which implies $(a \otimes b)^* \leq (a^* \otimes b^*)$. \square

Remark 13 (construction of hedges via substructures). Let $\mathbb{G} := (\mathcal{G}, *, id)$ be an action network with $\mathcal{G} := (V, E, \varrho)$. We say that a subset D of E forms a *substructure* of \mathbb{G} if $idV \subseteq D$ and $c * d \in D$ holds for all $(c, d) \in D^{(2)}$. If $\mathfrak{R} := (R, +, \otimes, 0, 1, \sum)$ is a complete semiring and D forms a substructure of \mathbb{G} , then the set

$$U := \{u \in S^E \mid ue = 0 \text{ for all } e \in E \setminus D\}$$

forms a substructure of the complete semiring $\mathfrak{R}[\mathbb{G}]$. In case \mathfrak{R} is a join complete semiring, the hedge associated with U is given by

$$* : R^E \rightarrow R^E, u \mapsto u_D$$

where $u_{De} := ue$ for all $e \in D$ and $u_{De} := 0$ for all $e \in E \setminus D$.

One application of the above for mathematical morphology is when the action network is given by \mathbb{Z}_{add}^2 and the substructure is $D := 2\mathbb{Z}^2$, where the join complete semiring \mathfrak{R} is induced by a t-norm.

Finally we present a generalization of the concept of hedges.

Proposition 6. *Let $\otimes : P_1 \times P_2 \rightarrow P$ be a biresiduation and let $(^{*1}, ^{*2}, *)$ be a triple of kernel operators. The following are equivalent*

1. $\forall p_1 \in P_1, \forall p \in P : (p_1 \xrightarrow{\otimes} p)^{*2} \leq p_1^{*1} \xrightarrow{\otimes} p^*$
2. $\forall p_1 \in P_1, \forall p_2 \in P_2 : p_1^{*1} \otimes p_2^{*2} \leq (p_1 \otimes p_2)^*$
3. $P_1^{*1} \otimes P_2^{*2} \subseteq P^*$

A star system is a triple of kernel operators where one of the above conditions holds.

Proof. “1 \Rightarrow 2”:

$$\begin{aligned} & p_1 \otimes p_2 \leq p_1 \otimes p_2 \\ \iff & p_2 \leq (p_1 \xrightarrow{\otimes} (p_1 \otimes p_2)) \\ \implies & p_2^{*2} \leq (p_1 \xrightarrow{\otimes} (p_1 \otimes p_2))^{*2} \\ \implies & p_2^{*2} \leq (p_1^{*1} \xrightarrow{\otimes} (p_1 \otimes p_2)^*) \\ \implies & p_1^{*1} \otimes p_2^{*2} \leq (p_1 \otimes p_2)^* \end{aligned}$$

“2 \Rightarrow 1”:

$$\begin{aligned} & (p_1 \xrightarrow{\otimes} p) \leq (p_1 \xrightarrow{\otimes} p) \\ \iff & (p_1 \otimes (p_1 \xrightarrow{\otimes} p)) \leq p \\ \implies & (p_1 \otimes (p_1 \xrightarrow{\otimes} p))^* \leq p^* \\ \implies & (p_1^{*1} \otimes (p_1 \xrightarrow{\otimes} p)^{*2}) \leq p^* \\ \iff & (p_1 \xrightarrow{\otimes} p)^{*2} \leq (p_1^{*1} \xrightarrow{\otimes} p^*) \end{aligned}$$

“2 \Rightarrow 3”: By 2, $p_1^{*1} \otimes p_2^{*2} \leq (p_1^{*1} \otimes p_2^{*2})^*$ and since $*$ is a kernel operator we have equality and therefore $p_1^{*1} \otimes p_2^{*2} \in P^*$.

“3 \Rightarrow 2”: Since $p_1^{*1} \otimes p_2^{*2} \leq p_1 \otimes p_2$ and $p_1^{*1} \otimes p_2^{*2} \in P^*$ we have $p_1^{*1} \otimes p_2^{*2} \leq (p_1 \otimes p_2)^*$. \square

The previous proposition gives rise to the following application.

Proposition 7. *Let $\mathfrak{L} := (\mathbb{L}_1, \mathbb{L}_2, \mathbb{L})$, $\mathfrak{P} := (\mathbb{P}_1, \mathbb{P}_2, \mathbb{P})$ be triples of complete lattices and let $\otimes_{\mathfrak{L}} : L_1 \times L_2 \rightarrow L$ and $\otimes_{\mathfrak{P}} : P_1 \times P_2 \rightarrow P$ be biresiduations. Then for every morphism $\Phi := (\varphi_1, \varphi_2, \varphi)$ from $(\mathcal{A}(\mathfrak{L}), \otimes_{\mathfrak{L}})$ to $(\mathcal{A}(\mathfrak{P}), \otimes_{\mathfrak{P}})$ the triple $(\varphi_1 \circ \varphi_1^+, \varphi_2 \circ \varphi_2^+, \varphi \circ \varphi^+)$ forms a star system w.r.t. $(\mathfrak{P}, \otimes_{\mathfrak{P}})$.*

Proof. It suffices to verify property 3 from the previous proposition:

$$(\varphi_1 \circ \varphi_1^+)P_1 \otimes_{\mathfrak{P}} (\varphi_2 \circ \varphi_2^+)P_2 = \varphi(\varphi_1^+P_1 \otimes_{\mathfrak{L}} \varphi_2^+P_2) \subseteq \varphi L = (\varphi \circ \varphi^+)P.$$

□

As a consequence of our considerations we receive the following extension of Theorem 3.

Theorem 4. Let \otimes be a biresiduation on a triple of complete lattices $\mathfrak{L} := (\mathbb{L}_1, \mathbb{L}_2, \mathbb{L})$. Then the star systems w.r.t. (\mathfrak{L}, \otimes) are in one-to-one correspondence with the substructures of $(\mathcal{A}(\mathfrak{L}), \otimes)$.

9 Conclusion

Some background information: Our paper is indeed based on our 2012 paper "A Macroscopic Approach to FCA and its Various Fuzzifications" but goes far beyond it. Who carefully reads our present paper will realize that notions have been refined and adjusted to our situation. Roughly said, biresiduation generalizes residuated posets while biadditivity generalizes complete semirings. What we mainly need is a specialization where these two concepts meet, that is, residuated complete lattices (algebraic logic point of view) which correspond to join complete semirings (linear algebra point of view). The importance of the latter point is for constructions.

Looking into Isabelle Bloch's constructions for mathematical morphology, we found out that there is a general framework in linear algebra over complete semirings which is outlined in section 5, named "Construction of Convolution Algebras". These turn out to be again complete semirings. So the bijective correspondence between residuated complete lattices and join complete semirings can be lifted from a "coordinate level" to a "space level". That is what Bloch essentially does in a special situation (without putting this into a general framework) and we derive from the general construction of convolution algebras over join complete semirings (and a subsequent paradigm shift into residuated complete lattices).

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