Modification of Fourier Approximation for Solving Boundary Value Problems Having Singularities of Boundary Layer Type

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Abstract. A method for approximating smooth functions has been developed using non-polynomial basis obtained by mapping of Fourier series domain to the segment $[-1, 1]$. High rate of convergence and stability of the method is justified theoretically for four types of coordinate mappings, the dependencies of approximation error on values of derivatives of approximated functions are obtained. Algorithms for expanding of functions into series with coupled basis composed of Chebyshev polynomials and designed non-polynomial functions are implemented. It was shown that for functions having high order of smoothness and extremely steep gradients in the vicinity of bounds of segment the accuracy of proposed method cardinally exceeds that of Chebyshev's approximation. For such functions method allows to reach an acceptable accuracy using only $N = 10$ basis elements (relative error does not exceed 1 per cent)

Keywords: singular perturbation, small parameter, coordinate mapping, boundary value problem, Fourier series, Chebyshev polynomial, non-polynomial basis, estimate of convergence rate, collocation method

1 Introduction

By now, a huge amount of urgent scientific and technological problems reduces to boundary value problems for differential equations having pronounced singularities of boundary layer type. The most popular approaches to solving them are based on construction of computational grids with piecewise linear/polynomial approximation of the unknown function in each cell of grid [1]. Such approaches provide relatively low rate of convergence and lead to essential refinement of grid in the vicinity of boundary layer and consequently to growth of computational costs and errors. In [2,3] methods of coordinate transformations were developed which allow to decrease the influence of mentioned effect due to application of special coordinate mappings eliminating the singularity. Nevertheless, the analysis of key issue concerning the smoothness of such transformations and its influence on the quality of approximation method is absent. Frequently, authors restrict their self by transforming uniform or more special grid and performing numerical experiments using finite difference or spectral methods [2]– [5].

In the present paper a step aside from the traditional grid approaches is made and the approximations based on mapping of Fourier series domain to the segment $[-1, 1]$ are used to approximate functions having singularities of boundary layer type. One of such mappings assigned by function $\cos(x)$ transforms Fourier basis to basis composed of Chebysev polynomials. A special feature inherent to these bases is the absence of saturation of corresponding approximations [6]. It means that using Fourier and Chebyshev bases allows to obtain the asymptotic of error of best polynomial approximation while approximating functions having any order of smoothness or singularities in complex plain. The loss of effectiveness of the mentioned approximations while solving problems having singularities of boundary layer type is caused by degradation of asymptotical properties of best polynomial approximations with fast increase of gradients of approximated function. In order to eliminate this problem, trigonometric Fourier basis can be transformed into non-polynomial algebraic one retaining all its good properties of high convergence rate and computational stability, but specially adapted to approximation of smooth functions having singularities of boundary layer type.

2 Preliminary information. Problem description

In framework of approximation theory of continuous and smooth functions $f(x)$ $(x \in D \subset \mathbb{R})$ that are elements of spaces $C(D)$, $W_p^r(M, D) = \{f \in C^r(D) : |f^{(r)}| \leq M = M(r)\}$ the following notions are often used (see for example [6]).

1. **Norms of function** $\|f\| = \max\limits_{x \in D} |f|$, $\|f\|_p = \left(\int\limits_D f^p(x)dx \right)^{\frac{1}{p}}$.

2. **Finite-dimensional approximating space** K_n with elements used for approximation of $f(x)$ that usually are series or polynomials including n summands or monomials.

3. **Operator of approximation (or simply approximation) of function** $f(x)$ is a continues mapping P_n performing a projection of functional space on approximating one ($P_n : C(D) \to K_n$ or $P_n : W_p^r(M, D) \to K_n$).

4. **Best approximation of function** $f \in C(D)$ is element $e_n(f, K_n) \in K_n$ providing the lower bound to be reached $\varepsilon_n^*(f, K_n) = = \varepsilon_n^*(f) = \inf\limits_{g \in K_n} \|f - g\|$.

5. **Method without saturation** (loose definition) is a method of approximation of function $f(x)$ having asymptotic of error of the best polynomial approximation for any order of smoothness of $f(x)$. Rigorous definition based on the analysis of asymptotic of Alexandrov's diameters is given in [6].

The results obtained in works by Lebegue, Faber, Jackson, Bernstain allow to separate three classes of smooth functions with fundamentally different asymptotical behavior of errors of best approximations in space K_n of algebraic polynomials of nth power. The similar asymptotical behavior is valid for periodic functions and trigonometrical polynomials

I. If $f(x) \in W_p^r(M,D)$ is r-times continuously differentiable function on segment D and all its derivatives up to order r are bounder by value $M(r)$, then

$$\sup_{f\in W_p^r(M,D)} \varepsilon_n^*(f,K_n) \le M(r)C_r n^{-r}, \tag{1}$$

where C_r depends on r only, [7].

II. If $f(x) \in C^\infty(D)$ is infinite differentiable function with a singularity (like pole) on complex plain, then one can find a number q $(0 < q < 1)$ and a sequence of polynomials $P_n(x)$, such that

$$\|f(x) - P_n(x)\| \le Cq^n,\ x \in D, \tag{2}$$

here $q < 1$ is defined by location of singularity in complex plain, C is constant, [8].

III. If $f(x) \in Ent$ is entire function, then

$$\varepsilon_n^*(f,K_n) \le \frac{M(n)(\mathrm{diam}D)^n 2^{1-2n}}{n!}, \tag{3}$$

where $M(n) = \|f^{(n)}\| = o(n!)$. It follows from estimates of error of polynomial interpolation (see [9]) and Cauchy–Hadamard inequality.

Remark 1. For smooth functions of I and III classes the accuracy of best approximations depends on maximal values of derivatives of function on D (values of $M(r)$ and $M(n)$ in (1), (3)). For functions of class II the error can be defined through the values of function itself beyond the segment D on complex plain.

In order to implement the properties of best approximations in this work the Fourier and Chebyshev series are used. Note that such approximations are equal in a specific sense. Indeed, let $f \in C([-1,1])$, then performing the change of variable $x = \cos\theta$, $\theta \in [0,2\pi]$ one can obtain 2π-periodic even function $\tilde{f}(\theta) = f(\cos\theta)$. Fourier decomposition of it is $\tilde{f}(\theta) = \sum\limits_{k=0}^{\infty} a_k \cos(k\theta)$. Hence

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(k \arccos(x)) = \sum_{k=0}^{\infty} a_k T_k(x). \tag{4}$$

In other words Chebyshev polynomials $T_k(x)$ can by obtained by mapping Fourier series domain to the segment $[-1,1]$, see [10]. In [6] is proved that such approximations presents the methods without saturation and therefore they are extremely efficient for solving problems with smooth solutions. However, if values $M(r)$, C, $M(n)$ in (1)–(3) are large, then it can be wrong.

A simple example is a boundary-value problem for differential equation of second order with small factor ε of second derivative

$$\varepsilon\frac{d^2 f}{dx^2} - f = \varepsilon g''(x) - g(x),\ f(-1) = 1, f(1) = -1, \tag{5}$$

here $x \in [-1,1]$, $g(x)$ is a given smooth function. Solution to the problem is

$$f(x) = \xi(x) + g(x),\ \xi(x) = C_1 e^{A(0.5x+0.5)} + C_2 e^{-A(0.5x+0.5)},$$

$$C_1 = \frac{1+e^{-A}}{e^{-A}-e^{A}},\ C_2 = \frac{1+e^{A}}{e^{A}-e^{-A}}.$$

Here $\xi(x)$ is an exponential boundary value component of $f(x)$, C_1, C_2, $A = \sqrt{1/\varepsilon}$ are constants. Table 1 shows the values of dimension of K_n ensuring that the relative errors of approximation is never higher than 1 per cent. These results were obtained in accordance with the estimates of best polynomial approximations (1), (3), while function $g(x)$ has different orders of smoothness.

Table 1. Dimension n ensuring that relative error of approximation is not higher than 1 per cent

Order of smoothness	The value of small parameter ε $\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$
$r = 1$	146 430	1 170 100	19 537 000	219 560 000
$r = 2$	5 997	65 952	714 010	7 643 800
$r = 3$	2 265	24 251	256 530	2 691 200
$r = 4$	1 426	15 037	157 040	1 629 500
$r = 5$	1 090	11 390	118 000	1 215 900
entire function	136	1 359	13 591	135 910

Thus, it can be observed that the higher order of smoothness is, the less data is necessary for recovering solution with a desired accuracy. However, even in the case of infinitely differentiable function a space K_n of dimension of many thousands can be required to reach considerably low accuracy of 1 per cent. This effect is shown on Fig 1 where a graph of solution to a problem (5) is given with $g(x) \equiv 0$ (i.e. a graph of function $\xi(x)$) and logarithm of error of approximation of $\xi(x)$ in space of Chebyshev polynomials

$$\nu = \max_{x\in[-1,1]} |\xi(x) - \sum_{m=0}^{n-1} a_m T_m(x)|.$$

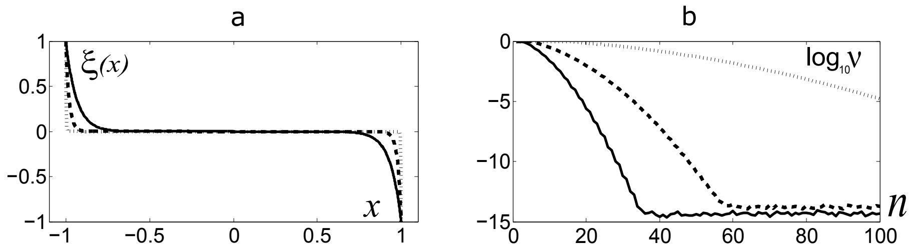

Fig. 1. Solution to the problem (5) with $\varepsilon = 10^{-3}$ (solid line), $\varepsilon = 10^{-4}$ (dash line), $\varepsilon = 10^{-6}$ (dote-and-dash line): a – graph of $\xi(x)$; b – dependance of $\log_{10}\nu$ on n

Fig. 1 shows that the values of n, given in Table 1 for function $\xi(x)$ are overestimated. This effect can be explained by concentration of Chebyshev nodes in the vicinity of boundary layers, see. [10], [11]. Considering a problem with boundary layer of size ρ having solution without singularities in inner part of the segment $[-1,1]$, to reach an accuracy of 1 per cent one should use a basis of

$$n \approx 3/\sqrt{\rho} \tag{6}$$

Chebyshev polynomials for approximation of unknown function. In the considered case $\rho \approx 5.6/A$ (here condition $\xi(\rho) = 0.1$ is used). Appearance of expression $\sqrt{\rho}$ in the denominator of fraction is caused by concentration of Chebyshev nodes. Indeed, uniformly distributed zeroes of trigonometrical monomials $\cos(k\theta)$ are concentrated in the vicinity of points ± 1 under the map $x = \cos(\theta)$ (see fig. 2 a). Moreover as $\cos(\theta) \sim 1 - \theta^2/2$ when $\theta \to 0$ and the first Chebyshev node $x_1 = \cos(\theta_1) = \cos\pi/2n$, then $|1 - x_1| \sim \pi^2/8n^2$. Further, the empirical requirement that even three nodes should lay on the boundary layer gives $3|1 - x_1| \leq \rho$ or $n \approx \dfrac{3\pi}{2\sqrt{2}\sqrt{\rho}}$ that corresponds to (6).

3 Description of a method

For efficient approximation of function having boundary layer component a modification of map $x = \cos(\theta)$ that transformed Fourier basis to Chebyshev one (4) is proposed. According to (6) a natural requirement is to use stronger concentration of zeroes of basis functions in the vicinity of segment borders (see 2 b).

Let us assume that $D = [-1,1]$. Define a function $x = æ(y) : [-1,1] \to [-1,1]$ satisfying the following basic requirements:

1) function $æ(y)$ is bijective mapping, $æ(1) = 1$, $æ(-1) = -1$;
2) $æ(y)$ is an infinite differentiable or even entire function;
3) an inverse function $y = æ^{-1}(x)$ can be expressed in analytical form or easily computed;
4) a derivative $æ'(y)$ in the vicinity of points ± 1 is close to zero.

Note that concentration of zeroes of basis functions in the vicinity of segment borders can be obtained due to the last requirement. One of possible forms of function $æ(y)$ is given on fig 2 b.

For approximation of function $f(x)$, $x \in [-1,1]$ let us consider the expansion of 2π-periodic even function $f(æ(\cos\theta))$ ($\theta \in \mathbb{R}$) into Fourier series:

$$f(æ(y)) = f(æ(\cos\theta)) \approx \sum_{k=0}^{n-1} a_k \cos(k\theta). \tag{7}$$

As a result one obtains

$$f(x) \approx P_n(x) = \sum_{k=0}^{n-1} a_k \cos[k \arccos(æ^{-1}(x))]. \tag{8}$$

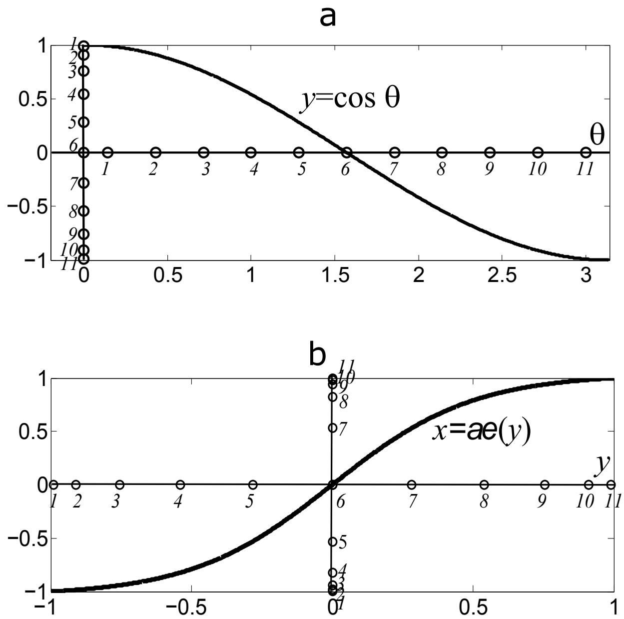

Fig. 2. Concentration of nodes of trigonometrical monomials $\cos(n\theta)$ as $n = 11$ (indicated by arrows): a – using mapping $y = \cos(\theta)$, b – using mapping $x = æ(y)$

Here we denote $y = \cos\theta$. Thus, the basis of approximating space K_n can be specified as $B_k(x) = \cos[k \arccos(æ^{-1}(x))]$, where zeroes of $B_k(x)$ are $x_m^k = æ\left(\cos\dfrac{(2m+1)\pi}{2k}\right)$, $k, m = 0, ..., n-1$. Note, that under the properties 1, 3, $æ(y)$, $B_k(x)$ are bijective easily computed functions.

Lemma 1. *Approximation* (8) *is equivalent to expansion of function* $f(æ(y))$ *into series with basis consists of Chebyshev polynomials* $T_k(y) = \cos(k \arccos(y))$*, that is why*

1) *for approximations* (8) *error estimations of best approximations* (1)–(3) *hold;*
2) *elements of matrices* $\mathfrak{B}_n$ *–* $b_{km} = B_k(x_m^n)$*,* $k, m = 0, ..., n-1$ *do not depend on* æ*.*

The proof of Lemma 1 is obvious taking into account (8) and that Chebyshev and Fourier expansions are approximations without saturation. The second condition of Lemma 1 concerns numerical approximation of f by collocation method with nodes x_m^n. Lemma declares that such a method is universal, its properties do not depend on the choice of function $æ(y)$.

To settle a key question on rate of growth of coefficients $M(r)$, $M(n)$ in estimates (1), (3) the following property was proved.

Lemma 2. *For derivative of composed function the followin equality is valid*

$$[f(æ(y))]^{(n)} = \sum_{k=1}^{n} f^{(k)}(æ(y)) \Bigg[\sum_{\substack{\mathcal{A}_{kl}=(\alpha_1,\ldots,\alpha_k) \\ \alpha_1+\ldots+\alpha_k=n}} C_{kl}^{n} \prod_{j=1}^{k} (æ(y))^{(\alpha_j)} \Bigg], \tag{9}$$

where the second summation goes over all possible integer partitions of number n – $\mathcal{A}_{kl}$, *that consist of* k *positive integer numbers* $\alpha_1,\ldots,\alpha_k$, $l = 1, \ldots, L$; C_{kl}^{n} *are constants. Moreover* $\forall n$ *as* $k = 1$ *and* $k = n$ *one has:* $L = 1$, $C_{11}^{n} = 1$, $C_{n1}^{n} = 1$.

This Lemma corresponds to the well-known Fa di Bruno's formula and can be proved using mathematical induction technique.

4 Analysis of four types of function $æ(y)$.

Now let us propose and investigate four types of function $æ(y)$. To this end the following notations are necessary. Let $D_a \in D$, $D_b \in D$ be neighborhoods of boundaries of segment D representing boundary layers, D_0 be a certain neighborhood of central point of D. Let us denote by $\|\cdot\|_L$, $\|\cdot\|_D$ the supremum norms of continues function on $D_a \cup D_b$ and on D_0 correspondingly. Further, we assume $a = -1$, $b = 1$, $\forall s < r$, $A_s\|f^{(s)}\| = \|f^{(s+1)}\|$, where $A_s > 1$ is constant, r is order of smoothness of $f(x)$. Typically for problems with boundary layer one has $A_s >> 1$. Let $A = \max\limits_{s<r} A_s$, $\rho(A)$ be a size of boundary layer (as it follows from the example with exponential boundary layer, ρ depends on A, see comments to formula (6)), $o(f^{(s)})$ be such value that $\lim\limits_{A_1,\ldots,A_{r-1}\to\infty} \dfrac{o(f^{(s)})}{\|f^{(s)}\|_L} = 0$.

I. Trigonometric function

$$æ(y) = \sin\left(\frac{\pi y}{2}\right). \tag{10}$$

Theorem 1. *Let* $\rho^2(A)A \to 0$ *as* $A \to \infty$. *For the approximation* (8) *with function* $æ(y)$ *of form* (10) *the estimate* (1) *is valid, but constant*

$M(r) = C_{r/2\ l}^{r}(\pi/2)^r\|f^{(r/2)}\| + o(f^{(r/2)})$ *(if* r *is even),*

$M(r) = C_{[r/2]l}^{r}\rho(A)(\pi/2)^r\|f^{([r/2])}\| + o(f^{([r/2])})$ *(if* r *is odd),*

where $[s]$ *denotes integer part of number* s, $C_{r/2\ l}^{r}$, $C_{[r/2]l}^{r}$ *are coefficients of* (9), *corresponding to integer partitions* $\mathcal{A}_{r/2\ l} = (\underbrace{2, \ldots, 2}_{r/2})$, $\mathcal{A}_{[r/2]\ l} = (\underbrace{2, \ldots, 2}_{[r/2]}, 1)$.

II. Polynomial of third power

$$æ(y) = ay^3 + by^2 + cy + d. \tag{11}$$

After taking into account properties 1)–4) of $æ(y)$-function one obtains $b = d = 0$, $a = 1 - c$, $1 \le c \le 1.5$. Assuming $p = c$, one gets

$$æ(y) = (1 - p)y^3 + py, \tag{12}$$

where $1 \leq p \leq 1.5$ is free parameter equal to the value of derivative æ$'(0)$ and determining the value æ$'(\pm 1)$: as $p \to 1.5$ æ$'(\pm 1) \to 0$.

Theorem 2. *For approximation* (8) *with function* æ(y) *of form* (12) *as* $p = 1.5$ *and* $\rho^2(A)A \to 0$ *as* $A \to \infty$ *the estimate* (1) *is valid, but constant*
$M(r) = C^r_{r/2\ l} 3^{r/2} \|f^{(r/2)}\| + o(f^{(r/2)})$ *(if* r *is even),*
$M(r) = C^r_{[r/2]l} \rho(A) 3^{[r/2]+1} \|f^{([r/2])}\| + o(f^{([r/2])})$ *(if* r *is odd),*
where $C^r_{r/2\ l}$, $C^r_{[r/2]l}$ *are the same as in theorem 1.*

Theorems 1,2 can be proved using Lemmas 1,2 and taking into account that values of first derivatives of considered æ-functions in points ± 1 vanishes. The component $\|f^{(r/2)}\|$ appears in the obtained expressions for $M(r)$ because the second derivative of æ-functions in points ± 1 are not equal to zero. This component grows much slower than $\|f^{(r)}\|$.

Remark 2. By changing in given theorems index "r" on index "n", one can obtain similar results for case of entire $f(x)$, namely the similar equalities for $M(n)$ from estimate (3) when æ(y) has form (10), (12).

Let us consider another approach to construction of æ(y)-function, it does not require first derivative of æ(y) to be equal to zero in points ± 1, but requires all the derivatives to be very small in vicinity of ± 1.

III. Function

$$æ(y) = \arctan(by)/\widetilde{b}, \ \widetilde{b} = \arctan b. \tag{13}$$

Theorem 3. *If in expression* (8) *function* æ(y) *of form* (13) *has parameter*

$$b << \sqrt[n+k-2]{\frac{\|f^{(k)}\|_L}{\|f^{(k)}\|_D}} = b_k, \ \forall k = 1, 2, ..., n, \tag{14}$$

then (8) *satisfies the estimate of accuracy of best approximation* (3) *with* $M(n)$ *equals for large* n *and* A *to the following*

$$M(n) = \max\left(\frac{\|f^{(n)}\|_L}{\widetilde{b}(\widetilde{b}b)^{n-1}}, \frac{\|f'\|_L}{\widetilde{b}} n!\right) + o(n!) + o(f^{(n)}). \tag{15}$$

Values

$$b \approx b_* = \frac{2}{\pi} \sqrt[n-1]{\frac{\|f^{(n)}\|_L}{\|f'\|_L} \frac{1}{n!}} \tag{16}$$

under condition $1.56 < b << \min\limits_{k=1,\dots,n} b_k$ *provide maximal rate of decrease of right part of* (3) *with growth of* n *and* $\widetilde{b}(\widetilde{b}b)^{n-1}$*-time profit in accuracy in comparison with the best polynomial approximation.*

IV. Function

$$æ(y) = \widetilde{\mu}\left(\frac{2}{1 + \exp(-\mu y)} - 1\right), \quad \widetilde{\mu} = \frac{1 + \exp(-\mu)}{1 - \exp(-\mu)}. \tag{17}$$

Theorem 4. *Let $f(x) \in W_p^r(M,D)$, $r > 1$ and in expression* (8) *function* $æ(y)$ *of form* (17) *has parameter, satisfying the inequalities*

$$\mu << \frac{1}{\alpha_s} W(\alpha_s \beta_s) = \mu_s, \ \forall s = 1, ..., r-1, \quad \mu << \pi \sqrt[r]{\frac{\|f^{(r)}\|_L}{\|f^{(r)}\|_D}} = \mu_0, \qquad (18)$$

where $W(x)$ is the Lambert W-function (the inverse function to $x = w\exp(w)$), $\alpha_s = \dfrac{s}{r-s}$, $\beta_s = 4^{\alpha_s}\pi \sqrt[r-s]{\dfrac{\|f^{(s+1)}\|_L}{\|f^{(s+1)}\|_D}}$. *In this case* (8) *satisfies the estimate of accuracy of best approximation* (1) *with $M(r)$ equals for large r and A to the following*

$$M(r) = \max\left(\|f^{(r)}\|_L (2\mu\widetilde{\mu}\exp(-\mu))^{r-1}, \ \|f'\|_L \widetilde{\mu} r! \right) + o(r!) + o(f^{(r)}). \qquad (19)$$

Values

$$\mu \approx \mu_* = -W\left(-\frac{1}{2} \sqrt[r-1]{\frac{\|f'\|_L r!}{\|f^{(r)}\|_L}} \right) \qquad (20)$$

under condition $1.57 < \mu << \min\limits_{s=0,1,\ldots,r-1} \mu_s$ *provide maximal rate of decrease of right part of* (1) *with growth of n and* $(\exp(\mu)/(2\mu\widetilde{\mu}))^{r-1}$*-time profit in accuracy in comparison with the best polynomial approximation. Here it is considered the branch of function $W(x)$ such that $W(x) \to -\infty$ as $x \uparrow 0$.*

Theorems 3,4 can be proved using Lemmas 1,2 and asymptotical analysis of growth of nth derivative of $f[æ(y)]$ with grows of n. These results can be extended to case of estimates (1), (3).

5 Computation of approximate values of functions with boundary value components

In this section one can find an experimental confirmation of results of theorems 1–4 considering approximation of solution to the boundary value problem (5) with exponential boundary layer. Detailed comments on correspondence of theoretical and experimental data given on fig. 3–5 and in Tables 2–5, are absent. Such correspondence becomes obvious if in theorems 1–4 one supposes the values $A_1, ..., A_{n-1}$ to be equal to the value of A from formula of solution $f(x)$ to the problem (5). This identification is correct because the values of derivatives $f^{(n)}(\pm 1)$ grow with a rate of A^n.

To implement the approximation according to (8) four considered types of function $æ(y)$ were used, expressions for inverse functions $æ^{-1}(x)$ were obtained and the values of zeroes of basis functions of approximations x_k^n were specified.

1. Sinus (*sin*), $\quad æ_s(y) = \sin\left(\dfrac{\pi y}{2}\right)$:

$$æ_s^{-1}(x) = \frac{2\arcsin x}{\pi}, \quad x_k^n = \sin\left[\frac{\pi\cos(\frac{(2k+1)\pi}{2n})}{2}\right], \ k = 0, ..., n-1.$$

2. Polynomial (pol), $\quad æ_p(y) = (1-p)y^3 + py$:

$$æ_p^{-1}(x) = R\left[-\cos\frac{\arccos z}{3} + \sqrt{3}\sin\frac{\arccos z}{3}\right],$$

$$R = \sqrt{\frac{p}{3(p-1)}}, \quad z = \frac{-3\sqrt{3}x\sqrt{p-1}}{2p^{3/2}},$$

$$x_k^n = (1-p)\cos^3\left[\frac{(2k+1)\pi}{2n}\right] + p\cos\left[\frac{(2k+1)\pi}{2n}\right], \ k = 0, ..., n-1.$$

These expressions were obtained using Cardano's method and analysis of three branches of inverse function $æ_p^{-1}(x)$.

3. Function inverse to tangents (tg), $\quad æ_t(y) = \dfrac{\arctan(by)}{\arctan(b)}$:

$$æ_t^{-1}(x) = \frac{\tan(x\arctan b)}{b}, \quad x_k^n = \frac{\arctan\left[b\cos(\frac{(2k+1)\pi}{2n})\right]}{\arctan(b)}, \ k = 0, ..., n-1.$$

4. Function including exponent (exp), $\quad æ_e(y) = \widetilde{\mu}\left[\dfrac{2}{1+\exp(-\mu y)} - 1\right]$:

$$æ_e^{-1}(x) = -\frac{1}{\mu}\ln\left[\frac{2}{\widetilde{\mu}x+1} - 1\right], \quad x_k^n = \widetilde{\mu}\left[\frac{2}{\exp\left\{-\mu\cos(\frac{(2k+1)\pi}{2n})\right\}+1} - 1\right],$$
$k = 0, ..., n-1$.

5.1 Approximations of boundary layer component $\xi(x)$

In this section results on numerical approximation of the solution $f(x)$ to the boundary-value problem (5) are given for different values of small parameter $\varepsilon = 10^{-3}, ..., 10^{-10}$. Let us consider first the case $g(x) \equiv 0$, then $f(x) = \xi(x)$ is an exponential boundary value component.

For searching the coefficients a_k of the expansion (8), where $k = 0, ..., n-1$ a system of n equations should be composed:

$$\sum_{j=0}^{n-1} a_j B_j(x_k^n) = f(x_k^n).$$

Matrix of this system with elements $\mathrm{b_{jk}} = B_j(x_k^n)$ is $C = (\mathrm{b_{jk}})$. Denoting column vectors

$$\alpha = (a_0, a_1, ..., a_N)^T, \quad \beta = (f(x_0), f(x_1), ..., f(x_N))^T,$$

one obtains a system of linear algebraic equations (SLAE) $C\alpha = \beta$. Finally the coefficients of expansion (8) can be expressed in form $\alpha = C^{-1}\beta$.

Remark 3. It was established by computations that the growth of number of basis function n does not affect the condition number of matrix C. The first five digits of it 1.4142 remain invariable while n grows from 1 to 200. This means that using the orthogonal transformations matrix C can be inverted on computer with high precision.

The algorithm of searching coefficients $a_k,\ k = 0, ..., n-1$ was implemented on MATLAB programming language for the following values of small parameter

$$\varepsilon = 10^{-3}, 10^{-4}, ..., 10^{-10}.$$

For each of considered types of æ(y) a grid of points $(z_1, z_2, ..., z_K)$ was generated on the segment $[-1; 1]$. It was used for searching the maximal deviation of a given function $f(x)$ form its approximation $P_n(x)$ by enumeration over all points. Due to the fact that function has boundary layer in the vicinity of points -1 and 1 a grid for enumeration was composed of large amount of Chebyshev nodes providing significant concentration of grid near the bounds of segment. The number of nodes $z_1, z_2, ..., z_K$ was chosen so that the doubling of it does not affect first 2–3 digits of error $\nu = \max\limits_{i=1,...,K} |f(z_i) - P_n(z_i)|$.

For example if $\varepsilon = 10^{-5}$ and function æ$_t(y)$ is used, the experiments show that for $n = 48$

$$\text{as } K = 100\ 000 \text{ point} \qquad \text{the error } \nu = 2.7737 \times 10^{-13}, \tag{21}$$

$$\text{as } K = 200\ 000 \text{ points} \qquad \text{the error } \nu = 2.7734 \times 10^{-13}. \tag{22}$$

Consequently, we conclude that number $K = 100\ 000$ is sufficient for estimating the error ν with desired accuracy.

Now let us use a single coordinate plain to represent dependencies of $\log_{10} \nu$ on the number of basis functions n of approximation P_n for different types of function æ(y) together with well known Chebyshev approximations. (see. fig. 3, 4). Number of points for enumeration K here is equal to a maximal one of all K obtained for each of considered types. Note that the values of parameters p, b, μ in these results are taken from the vicinity of their "optimal" values (see Table 2). Further method for searching these values is explained.

Table 2. "Optimal" values of parameters

Parameter	Value of small parameter ε							
of approximation	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
p	1.1	1.2	1.3	1.35	1.45	1.46	1.48	1.48
b	1.2	1.6	3.5	15	35	75	300	680
μ	1.3	2.5	3	4.5	5.5	6.8	8.2	9.4

On given figures one can observe that the decrease of values of small parameter ε results in fast decrease of convergence rate of Chebyshev expansions. Whereas, the methods designed using functions æ$_t(y)$, æ$_e(y)$ do not significantly

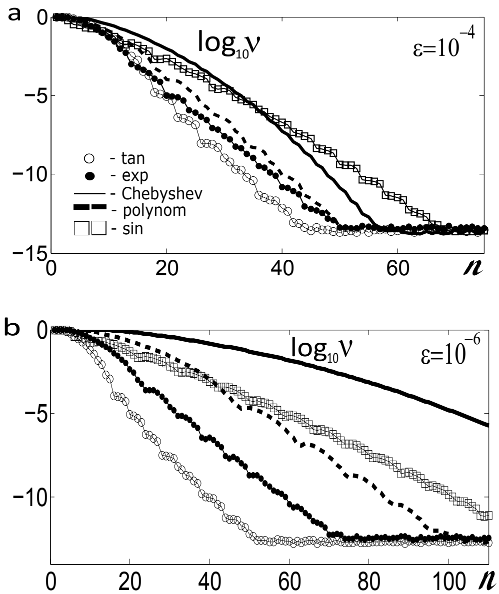

Fig. 3. Dependencies of $\log_{10}\nu$ on number n for values $\varepsilon = 10^{-4}$ (b), $\varepsilon = 10^{-6}$ (b)

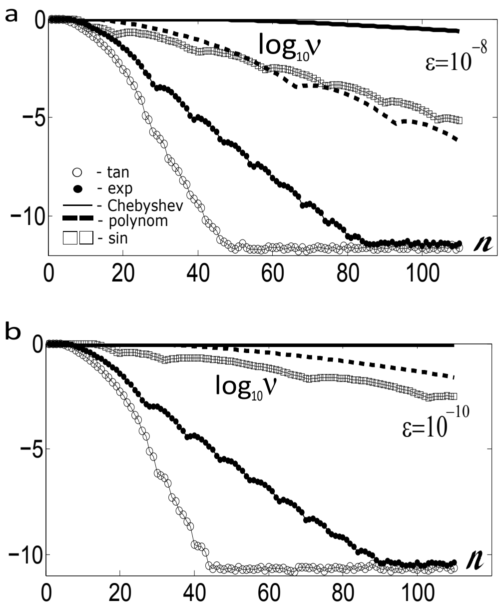

Fig. 4. Dependencies of $\log_{10}\nu$ on number n for values $\varepsilon = 10^{-8}$ (a), $\varepsilon = 10^{-10}$ (b)

change their convergence rate while decreasing small parameter. As it was already proved, increase of b, μ allows one to reduce the influence of steep gradient.

Now let us estimate "optimal" values of parameters p, b, μ for approximations based on æ$_p$, æ$_t$, æ$_e$. To this end the dependencies of logarithm of error $\log_{10} \nu$ on n and value of parameter of function æ(y) were represented in Cartesian coordinate system, see those for æ$_t$ on fig. 5, 6.

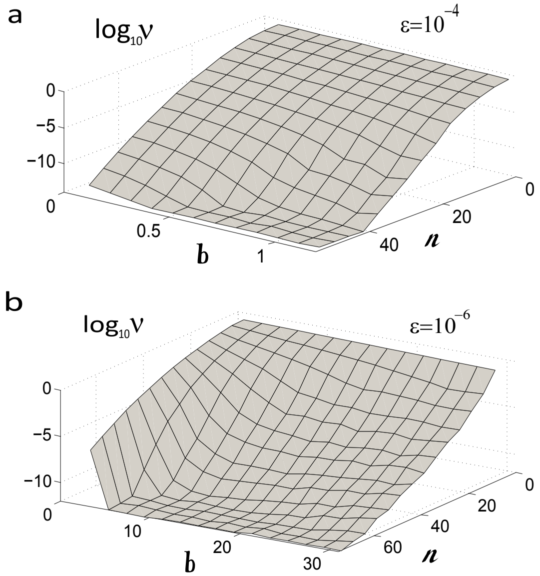

Fig. 5. Dependencies of value $\log_{10} \nu$ on the number n and the value of parameter b of method with function æ$_t$ as $\varepsilon = 10^{-4}$ (a), $\varepsilon = 10^{-6}$ (b), $\varepsilon = 10^{-8}$ (c), $\varepsilon = 10^{-10}$ (d)

The error was computed in a similar way as before, e.g. for æ$_t$ and $\varepsilon = 10^{-8}$ parameter b goes over all integer values from 0 to 100. Maximal convergence rate is provided by such a value $b = b^*$ that specifies maximal slope of a curve laying in section of the represented surface by a plain $b = b^* =$ const(or $p =$ const, or μ = const). For æ$_t$ and $\varepsilon = 10^{-8}$ $b^* \in [60; 80]$. This segment is marked with circle on the graph of fig. 6.

5.2 Approximation of solution of inhomogeneous problem (5)

Let us consider a function $f(x)$ that is solution to (5) with $g(x) = \sin(\pi x)$. Here $g(x)$ provides a perturbation in inner points of domain of problem. A question is how such perturbations can affect convergence of proposed methods.

$$\begin{gathered} f(x) = C_1 e^{A(\frac{1}{2}x+\frac{1}{2})} + C_2 e^{-A(\frac{1}{2}x+\frac{1}{2})} + \sin(\pi x), \\ C_1 = \frac{1+e^{-A}}{e^{-A}-e^{A}},\ C_2 = \frac{1+e^{A}}{e^{A}-e^{-A}},\ A = \sqrt{1/\varepsilon}. \end{gathered} \tag{23}$$

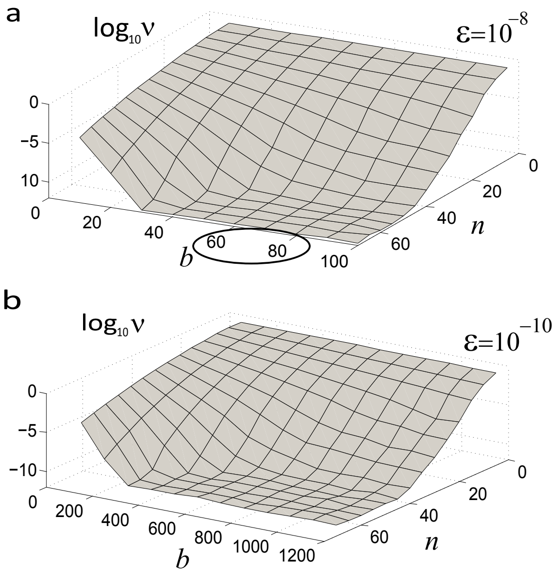

Fig. 6. Dependencies of value $\log_{10}\nu$ on the number n and the value of parameter b of method with function æ$_t$ as $\varepsilon = 10^{-8}$ (a), $\varepsilon = 10^{-10}$ (b)

Let ν be maximal values of deviation of approximation (8) from the values of function (23). ν was computed in numerical experiments by enumeration over points $z_1, ..., z_K$ as it was described, for different types of æ(y) and various values of n and ε: $\varepsilon = 10^{-3}, ..., 10^{-10}$. The values of parameters p, b, μ were taken from Table 2 excluding cases of small n. Numerical experiments showed that the smaller n is, the stronger appears the dependence of convergence rate on n.

While using functions æ$_s$, æ$_p$, æ$_e$ approximation errors for (23) agree with those obtained for $f(x) = \xi(x)$ (see fig 3, 4). "Optimal" values of parameters p and μ in these cases retain too. However the approximation based on æ$_t$ loose its convergence rate much faster. This effect is explained by inequality for b (see estimate (14)) that turns out to be more essential than, for example, (18). Even for moderate values of $\|f^{(s+1)}\|_D$ (in our case $\forall s\ \|f^{(s+1)}\|_D \approx 1$) one obtains that value of b should be considerably less than those given in Table 2. Then, according to (15), $M(n)$ becomes large and the convergence rate decreases.

In order to obtain efficient approximation with function æ$_t$ a coupled basis $B_k(x) \cup T_m(x)$ ($k = 0, ..., \mathcal{K}$, $m = 0, ..., M$, $\mathcal{K} + M = n - 2$) should be used. It is composed of designed functions $B_k(x)$ and Chebyshev polynomials $T_m(x)$:

$$f(x) \approx \sum_{k=0}^{\mathcal{K}} a_k B_k(x) + \sum_{m=0}^{M} c_m T_m(x),$$
$$T_m(x) = \cos(m \arccos(x)),\ B_k(x) = \cos\left(k \arccos\left[\tan(x\widetilde{b})/b\right]\right). \quad (24)$$

The idea of a method consists in two steps: first function $f(x)$ should be expand in basis $T_m(x)$ and then difference $f(x) - \sum_{m=0}^{M} c_m T_m(x)$ should be approximated using $B_k(x)$. Let us compose matrices

$$\Upsilon = (\gamma_{mi}),\ \gamma_{mi} = T_m(x_i),\ \text{where}\ x_i = \cos\left[\frac{(2i+1)\pi}{2(M+1)}\right],\ m, i = 0, ..., M;$$

$$\mathfrak{B} = (b_{kj}),\ b_{kj} = B_k(x_j),\ \text{where}\ x_j = \left(\arctan\left[b\cos(\frac{(2j+1)\pi}{2(\mathcal{K}+1)})\right]\right)/\widetilde{b},$$

$$k, j = 0, ..., \mathcal{K}.$$

Introducing the notations of vectors $\alpha = (a_0, ..., a_{\mathcal{K}})$, $\gamma = (c_0, ..., c_M)$, $f_\alpha = (f(x_0), ..., f(x_{\mathcal{K}}))$, $f_\gamma = (f(x_0), ..., f(x_M))$ one can found the coefficients of expansion in Chebysev basis $\gamma = \Upsilon^{-1} f_\gamma$. Further the following SLAE was composed

$$\mathfrak{B}\alpha = f_\alpha - C\gamma,$$

$$C_{jm} = T_m(x_j) = \cos\left(m \arccos\left[\arctan\left\{b\cos(\frac{(2j+1)\pi}{2(\mathcal{K}+1)})\right\}/\widetilde{b}\right]\right).$$

and the values of coefficients $a_0, ..., a_{\mathcal{K}}$ were obtained

$$\alpha = \mathfrak{B}^{-1}\left[f_\alpha - C\Upsilon^{-1} f_\gamma\right]. \tag{25}$$

The computations by formula (25) can be modified. So, for small parameters $\varepsilon = 10^{-4}, \varepsilon = 10^{-6}$ the domain of distribution of Chebyshev nodes in (24) was narrowed from segment $[-1; 1]$ to $[-0.85; 0.85]$ in case $\varepsilon = 10^{-4}$ and to $[-0.95; 0.95]$ in case $\varepsilon = 10^{-6}$.

Tables 3–5 show the value of approximation error ν obtained for function (23) using Chebyshev basis and the designed ones B_k with functions æ(y) of four considered types. If value of parameters of æ(y) differs from those given in Table 2 than it is explicitly indicated in brackets. So does a number of basis functions $B_k(x)$ plus number of basis polynomials $T_m(x)$ for approximations obtained using æ$_t$, see (24).

Table 3. Values of ν obtained for function (23) with $\varepsilon = 10^{-6}$

n	Chebyshev	$\sin(y)$	$pol(y)$	$\exp(y)$	$\tan(y)$
10	0.997	0.168	0.208	0.069(5.5)	0.036(60, 7 + 3)
20	0.727	0.047	0.064(1.5)	0.005	5.21×10^{-4}(80, 14 + 6)
30	0.359	0.007	0.025	2.4418×10^{-5}	7.935×10^{-6}(50, 23 + 7)
40	0.147	7.659×10^{-4}	0.001	3.679×10^{-7}	4.283×10^{-8}(14, 30 + 10)
50	0.051	5.419×10^{-5}	2.216×10^{-5}	1.936×10^{-9}	2.637×10^{-10}(14, 38 + 12)
60	0.015	8.019×10^{-6}	5.348×10^{-7}	3.738×10^{-11}	1.436×10^{-12}(9, 47 + 13)
70	0.004	7.348×10^{-7}	2.533×10^{-8}	5.473×10^{-13}	2.625×10^{-13}(9, 57 + 13)
80	7×10^{-4}	3.346×10^{-8}	7.458×10^{-10}	3.321×10^{-13}	2.26×10^{-13}(9, 67 + 13)
90	1.2×10^{-4}	2.941×10^{-9}	1.791×10^{-11}	3.375×10^{-13}	2.597×10^{-13}(9, 77 + 13)

Table 4. Values of ν obtained for function (23) with $\varepsilon = 10^{-8}$

n	Chebyshev	$\sin(y)$	$pol(y)$	$\exp(y)$	$\tan(y)$
10	1.0000	0.5024	0.5903(1.5)	0.2040(7.8)	$0.0463(650, 7+3)$
20	1.0000	0.1972	0.6521	0.0366	$6.0843 \times 10^{-4}(180, 14+6)$
30	0.9987	0.1119	0.2743	3.5359×10^{-4}	$1.2506 \times 10^{-5}(140, 23+7)$
40	0.9730	0.0208	0.0873	1.0721×10^{-5}	$5.3592 \times 10^{-8}(110, 30+10)$
50	0.8920	0.0130	0.0186	3.8726×10^{-7}	$4.9031 \times 10^{-10}(90, 39+11)$
60	0.7705	0.0032	0.0020	8.8276×10^{-9}	$3.4193 \times 10^{-12}(90, 47+13)$
70	0.6384	0.0011	4.28×10^{-4}	4.0243×10^{-10}	$1.8532 \times 10^{-12}(90, 57+13)$
80	0.5144	3.7×10^{-4}	1.42×10^{-4}	9.3578×10^{-12}	$1.9960 \times 10^{-12}(90, 67+13)$
90	0.4059	5.7×10^{-5}	1.27×10^{-5}	4.3484×10^{-12}	$2.3292 \times 10^{-12}(90, 77+13)$

Table 5. Values of ν obtained for function (23) with $\varepsilon = 10^{-10}$

n	Chebyshev	$\sin(y)$	$pol(y)$	$\exp(y)$	$\tan(y)$
10	1.0000	1.0005	1.0003	0.3875(10.4)	$0.0529(6500, 7+3)$
20	1.0000	0.3498	0.9986	0.0391	$4.7344 \times 10^{-4}(4000, 13+7)$
30	1.0000	0.2231	0.9282	0.0027	$1.0203 \times 10^{-5}(1200, 22+8)$
40	1.0000	0.2041	0.7389	2.1276×10^{-4}	$9.2063 \times 10^{-8}(1100, 31+9)$
50	1.0000	0.1418	0.5336	1.5681×10^{-5}	$3.5014 \times 10^{-10}(1100, 38+12)$
60	1.0000	0.0700	0.3642	1.0989×10^{-6}	$2.1094 \times 10^{-11}(1100, 47+13)$
70	1.0000	0.0226	0.2371	7.3964×10^{-8}	$2.2963 \times 10^{-11}(1100, 57+13)$
80	0.9999	0.0223	0.1470	4.8155×10^{-9}	$1.8534 \times 10^{-11}(1100, 67+13)$
90	0.9994	0.0124	0.0865	3.0489×10^{-10}	$2.1706 \times 10^{-11}(1100, 77+13)$
100	0.9973	0.0041	0.0481	4.0388×10^{-11}	$2.1034 \times 10^{-11}(1100, 87+13)$

6 Conclusion

In this work a method for approximating smooth functions having boundary layer components was developed, justified and implemented. It is based on expansion of function into series with basis consisting of non-polynomial functions obtained from trigonometric Fourier one by special mapping operation. Such representation ensures the estimations of accuracy of best polynomial approximations, but essentially reduces the values of coefficients in them. That is why convergence can be observed starting from small number of basis elements. Moreover the proposed approximations have good properties of numerical stability inherent to Fourier expansions.

Further development and successful application of the method is connected with analysis of different forms of æ-function. Proposed method can be modified for approximation of functions having singularity in inner point of domain, or even for problems with unknown position of singularity. From the other side, combination of these approximations with collocation methods will allow to design efficient algorithms for solving singularly-perturbed boundary value problems for differential equations.

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