Optimal Control of Human Capital Allocation for Piecewise Smooth Dynamic Model

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Abstract. The article presents optimal control of human capital allocation under the condition of smoothness abnormality in the right-hand member of the dynamic system. It describes an individual's behavior who strives to leave to his/her heirs the maximum of cash and tries to get the maximum utility through controlling expenditures and time consumption for work and studies. The objective function of the model maximizes the human life utility. The latter is formed by the human capital development and financial savings for heirs. The authors developed special methods how to find optimal control and take into account smoothness abnormality in the right-hand member of the dynamic system. They defined the optimality conditions and designed algorithmic support and software to find optimal control and interpret its content.

Keywords. Optimal control · human capital · maximum principle · piecewise smooth dynamic system.

1 Introduction

Nowadays the design and development of new technologies, computing and informatics are becoming of particular importance to all spheres of social life. Because of that workforce requires excellent qualifications and specific knowledge. In this regard we can state that human capital is the key factor for society development. Besides the modeling in human capital allocation as well as its management to achieve the maximum results are the topical modern issues to study.

Human capital is the combination of physical and mental capacities of a person, his/her knowledge, abilities and skills, life and professional experience used in public reproduction [1]. The acquisition of human capital is made up from the investments into health, safety, life quality, science, education and culture. One of the major factors leading to human capital growth is the investment in education and development of employees' professional qualifications. And the investment to maintain healthy lifestyle provides the increase in the use duration of human capital and the improvement of its quality.

As time goes on, investments into human capital are to provide incomes to its holder. Though spending part of his/her employment time on education, a person cannot count on the full volume of potential salary. Thus there is a challenge to allocate a reasonable portion of time for education.

To respond to this challenge let's apply K. Pohmer model designed for the optimal management of personal income allocation [2].

2 Mathematical Model of Human Capital Allocation and Its Numerical Solution

Let's assume that at the life beginning any individual has an original human capital H_0, i.e. innate abilities, health, etc., and an original financial capital K_0, i.e. financial funds of parents, a state, etc.

$$H(0) = H_0 ,\ K(0) = K_0 . \tag{1}$$

Human capital changes as the result of education attainment, life and professional experience acquisition. Let's consider the function of human capital raising:

$$F\left[H(t), s(t), l(t)\right], \tag{2}$$

where

$H(t)$ is human capital;

$l(t)$ is the fraction of total time spent for working;

$s(t)$ is the amount of employment time spent for education and professional expertise development.

Note that as the time goes, knowledge becomes less relevant and thus loses its value . The changing of human capital will look like

$$\dot{H}(t) = F\left[H(t), s(t), l(t)\right] - \delta H(t) , \tag{3}$$

where δ is the loss index of acquired skills, it equals to amortization norm for fixed capital.

Human capital affects financial capital and we can write the rate of financial capital change as

$$\dot{K}(t) = iK(t) + rH(t)g(s(t))l(t) - c(t) , \tag{4}$$

where

$K(t)$ is the financial capital;

i is the capital cost factor;

r is the price per human capital unit;

$c(t)$ is the consumption;

$g(s)$ is the potential salary in relation to education level which is subject to the time spent.

$g(s)$ is the production capability curve that defines the correlation between education and earnings: the lower (higher) is s, the more (less) is potential salary though the level of human capital acquisitions is lower (higher). Thus $g(s)$ satisfies the conditions:

$$g(0)=1,\ g(1)=0,\ g_s<0,\ g_{ss}<0. \tag{5}$$

Derived criteria (5) satisfy to the function

$$g(s)=-as^2+bs+c,\ a>0. \tag{6}$$

Applying to conditions (6) we will define the required function

$$g(s)=1-(1-a)s-as^2,\ a>0. \tag{7}$$

So the model for human capital distribution has two processes

$$\dot{H}(t)=F[H(t), s(t), l(t)]-\delta H(t), \tag{8}$$

$$\dot{K}(t)=iK(t)+rH(t)g(s(t))l(t)-c(t), \tag{9}$$

with initial conditions

$$H(0)=H_0,\ K(0)=K_0, \tag{10}$$

and restrictions imposed on controllable variables

$$c(t)>0,\ 0\le l(t)<1,\ 0\le s(t)\le 1. \tag{11}$$

Under restrictions we mean a constant consumption of goods and the poverty of employment time, for example, retirement period. The employment time is to be less than 1, as an individual needs to have a rest. The fraction of total time spent for a rest, sleeping included is ($1-l(t)$).

One of the priorities for any individual is to give birth to healthy off-springs and sustain their growth. To achieve these goals a person is to increase his/her human capital and raise financial capital for his/her heirs. So the objective function of the model maximizes human life utility made up of human capital development and financial savings for heirs.

$$I(c, s, l)=\int_0^T U(t, c, l, H)e^{-\rho t}dt+Z[K(T)]\to \max, \tag{12}$$

where
T is the duration of an individual's life;
ρ is the time preference rate of an individual;

$U(t, c, l, H)$ is the utility function dependent on consumption, employment time and human capital.

$Z[K(T)]$ is the utility function for heirs.

The functions $U(t, c, l, H)$ and $Z[K(T)]$ have a typical for economics definitions

$$U = \frac{c^{1-\alpha}}{1-\alpha} + \xi \frac{(1-l)^{1-\beta}}{1-\beta} + \nu t \frac{H^{1-\gamma}}{1-\gamma}, \tag{13}$$

$$Z = \Gamma \frac{K(T)^{1-k}}{1-k}, \tag{14}$$

where ratios α, β, γ and k point to utility elasticity and ξ, ν and Γ are the measures for rest, human capital and heirs capital and νt shows that with aging education plays a greater role.

Production function for human capital looks like

$$F = \sigma H^{\varepsilon} s l, \tag{15}$$

where σ is the parameter in the Cobb-Douglas function , ε is the partial elasticity of human capital.

So the optimal management problem looks like

$$I(c, s, l) = \int_0^T \left[\frac{c(t)^{1-\alpha}}{1-\alpha} + \xi \frac{(1-l(t))^{1-\beta}}{1-\beta} + \nu t \frac{H(t)^{1-\gamma}}{1-\gamma} \right] e^{-\rho t} dt + \Gamma \frac{K(T)^{1-k}}{1-k} \to \max, \tag{16}$$

$$\dot{H}(t) = \sigma H(t)^{\varepsilon} s(t) l(t) - \delta H(t), \tag{17}$$

$$\dot{K}(t) = iK(t) + rH(t)l(t)\left(1 - (1-a)s(t) - as(t)^2\right) - c(t), \tag{18}$$

$$H(0) = H_0, \ K(0) = K_0, \tag{19}$$

$$c(t) > 0, \ 0 \le s(t) \le 1, \ 0 \le l(t) < 1. \tag{20}$$

The problem to solve (16) – (20) is an optimal control problem with a free right end and control restrictions. Let's use the Pontryagin maximum principle [3].

$$\begin{aligned} \mathrm{H} = {} & \psi_1 \cdot \left(\sigma H^{\varepsilon} s l - \delta H\right) + \psi_2 \cdot \left(iK + rHg(s)l - c\right) + \\ & + \left(\frac{c^{1-\alpha}}{1-\alpha} + \xi \frac{(1-l)^{1-\beta}}{1-\beta} + \nu t \frac{H^{1-\gamma}}{1-\gamma} \right) \cdot e^{-\rho t}, \end{aligned} \tag{21}$$

where conjugate variables ψ_1 and ψ_2 satisfy to

$$\begin{cases} \dot{\psi}_1 = \psi_1 \cdot \left(\delta - \sigma\varepsilon H^{\varepsilon-1} sl\right) - \psi_2 rg(s) l - \nu t H^{-\gamma} e^{-\rho t}, \\ \psi_1(T) = 0, \\ \dot{\psi}_2 = -i\psi_2, \\ \psi_2(T) = \Gamma K(T)^{-k}. \end{cases} \tag{22}$$

Optimal control is defined when the Pontryagin function is at maximum

$$-\psi_2 \bar{c} + \frac{\bar{c}^{1-\alpha}}{1-\alpha} e^{-\rho t} = \max_{c>0} \left\{ -\psi_2 c + \frac{c^{1-\alpha}}{1-\alpha} e^{-\rho t} \right\}, \tag{23}$$

$$\begin{aligned} &\psi_1 \sigma H^{\varepsilon} \bar{l}\bar{s} + (a-1)\psi_2 r H \bar{l}\bar{s} - a\psi_2 r H \bar{l}\bar{s}^2 = \\ &= \max_{0 \le s \le 1} \left\{ \psi_1 \sigma H^{\varepsilon} ls + (a-1)\psi_2 r H ls - a\psi_2 r H l s^2 \right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\psi_1 \sigma H^{\varepsilon} s \bar{l} + \psi_2 r H g(s) \bar{l} + \xi \frac{\left(1-\bar{l}\right)^{1-\beta}}{1-\beta} e^{-\rho t} = \\ &= \max_{0 \le l < 1} \left\{ \psi_1 \sigma H^{\varepsilon} sl + \psi_2 r H g(s) l + \xi \frac{(1-l)^{1-\beta}}{1-\beta} e^{-\rho t} \right\}. \end{aligned} \tag{25}$$

Maximum of expressions (23) – (25) is possible when

$$\bar{c} = \left(\psi_2 e^{\rho t}\right)^{-\frac{1}{\alpha}}, \tag{26}$$

$$\bar{s} = \begin{cases} 0, & s^* < 0, \\ s^*, & s^* \in [0,1], \\ 1, & s^* > 1, \end{cases} \quad \text{where } s^* = \frac{a-1}{2a} + \frac{\psi_1 \sigma H^{\varepsilon-1}}{2a\psi_2 r}, \tag{27}$$

$$\bar{l} = \begin{cases} 0, & l^* \le 0, \\ l^*, & l^* > 0, \end{cases} \quad \text{where } l^* = 1 - \left(\frac{\psi_1 \sigma H^{\varepsilon} s + \psi_2 r H g(s)}{\xi} e^{\rho t} \right)^{-\frac{1}{\beta}}. \tag{28}$$

Considering the types of formulae received, let’s use numerical solution to find the optimal control.

To search an optimal path and optimal control we use software algorithm based on iterative method [3]. The value parameters of the problem are equal to the values in the table 1.

Table 1. The value parameters of the problem

$\alpha = 2$	$\beta = 1{,}5$	$\gamma = 0{,}8$
$\xi = 0{,}4$	$\nu = 0{,}0015$	$\Gamma = 0{,}8$
$\rho = 0{,}01$	$k = 0{,}8$	$\varepsilon = 0{,}35$
$\sigma = 1$	$r = 1$	$\delta = 0{,}01$
$a = 0{,}3$	$i = 0{,}04$	$T = 75$
$H_0 = 1$	$K_0 = 40$	

Figures 1 and 2 demonstrate the results we got after applying the software.

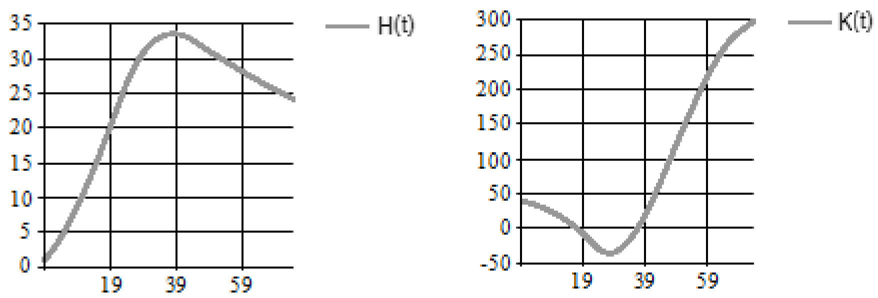

Fig. 1. Temporal dynamics of human capital and financial capital

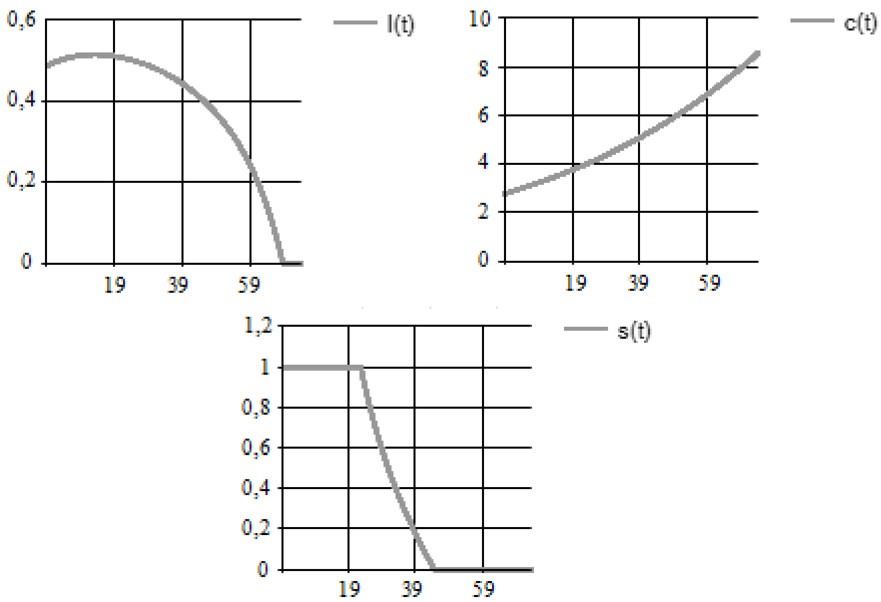

Fig. 2. Temporal variation of optimal control

The resulting graph of the control parameter "employment time" variation shows its maximum at the age of 14. It certifies the possibility of time saving as a young body needs less rest. And the graph for $s(t)$ parameter dynamics shows that all employment time is spent for education, both at school or university, and self-education as well. As the time goes, the amount of time for education is decreasing as an individual has accumulated some knowledge and tries to use it in his/her activities. The employment time decreases because of changes in personal life like marriages, birth of children. According to the statistical data an average person finishes his/her labor activity at the age of 70.

Under control parameters an individual's human capital grows by the age of 40. By that age an individual gets a certain professional expertise. Besides investments in education characterizing the period of negative capital at the age period between 18 and 37, and acquired experience provide stable income. The financial capital curve $K(t)$ starts to grow. The growth of consumption $c(t)$ follows the growth of an individual's income.

3 Optimal Control Problem for Discontinuous Dynamic System

Let's consider the right-hand smoothness abnormality in dynamic restrictions when i parameter characterizes capital cost factor and has two values depending on negative or positive sign of $K(t)$:

$$i = \begin{cases} i_1, & K(t) \geq 0, \\ i_2 > i_1, & K(t) < 0, \end{cases} \tag{29}$$

i.e. the problem is:

$$I(c, s, l) = \int_0^T \left[\frac{c^{1-\alpha}}{1-\alpha} + \xi \frac{(1-l)^{1-\beta}}{1-\beta} + \nu t \frac{H^{1-\gamma}}{1-\gamma} \right] e^{-\rho t} dt + \Gamma \frac{K(T)^{1-k}}{1-k} \to \max, \tag{30}$$

$$\dot{H}(t) = \sigma H^{\varepsilon} sl - \delta H\,, \tag{31}$$

$$\dot{K}(t) = \begin{cases} i_1 K + rHlg(s) - c, & K(t) \geq 0, \\ i_2 K + rHlg(s) - c, & K(t) < 0, \end{cases} \quad i_2 > i_1, \tag{32}$$

$$H(0) = H_0\,,\ K(0) = K_0\,, \tag{33}$$

$$c(t) > 0\,,\ 0 \leq s(t) \leq 1\,,\ 0 \leq l(t) < 1\,, \tag{34}$$

$$g(s) = 1 - (1-a)s - as^2\,.$$

In the optimization problem to solve (30) – (34) the system of differential equations is the system with the discontinuous right-hand part

$$\dot{x}(t)=\begin{cases} f_1(t,x(t),u(t)), & S(t,x)\ge 0,\\ f_2(t,x(t),u(t)), & S(t,x)<0,\end{cases}$$

where $x=(H,K)$ is the absolutely-continuous on the interval $[0,T]$ vector-function of phase, $u=(c,s,l)$ is the piecewise-continuous on the interval $[0,T]$ vector-function of phase. On the strength of all evidences the switching surface $S(t,x)$ is a continuously differentiable function that looks like $S(t,x)=K(t)$.

Let's consider the case of a single penetration of the switching surface by the path function at τ, where τ is the switching point, i.e. the point where $K(\tau)=0$.

To formulate the theorem on the necessary optimal conditions for optimal control we use the Pontryagin function:

$$\mathrm{H}(t,x,u,\psi(t))=\begin{cases} \mathrm{H}_1(t,x,u,\psi_1(t)), & S(x)\ge 0,\\ \mathrm{H}_2(t,x,u,\psi_2(t)), & S(x)<0,\end{cases}$$

where

$$\psi(t)=\begin{cases} \psi_1(t), & S(x)\ge 0,\\ \psi_2(t), & S(x)<0,\end{cases}$$

$\psi_1(t):[0,\tau)\to R^2$, $\psi_2(t):[\tau,T]\to R^2$,

$$\mathrm{H}_j(t,x,u,\psi(t))=\left[\frac{c^{1-\alpha}}{1-\alpha}+\xi\frac{(1-l)^{1-\beta}}{1-\beta}+\nu t\frac{H^{1-\gamma}}{1-\gamma}\right]e^{-\rho t}+\big(\psi_j(t),f_j\big(t,x(t),u(t)\big)\big),$$

$j=1,2$.

Theorem 1. The process $\overline{\omega}=(\overline{x}(t),\overline{u}(t),\tau)$, where τ is the switching point and which is optimal for the problem to solve (30) – (34). This condition is satisfied with the necessity of ψ_1, ψ_2 functions that are not simultaneously equal to zero and $\lambda_0\ge 0$ multiplier. Then the following conditions are fulfilled

1. optimal control $\overline{u}(t)=(\overline{c}(t),\overline{s}(t),\overline{l}(t))$, $t\in[0,T]$ in all continuity points affords a maximum of the Pontryagin function $\mathrm{H}_j(t,\overline{x}(t),u,\psi_j(t))$, $j=1,2$, in all allowable $u(t)$:

$$\mathrm{H}_j(t,\overline{x}(t),\overline{u}(t),\psi_j(t))=\max_{u(t)}\mathrm{H}_j(t,\overline{x}(t),u,\psi_j(t)),\ j=1,2;$$

2. the conjugate vector-function $\psi_j(t)=(\psi_j^1,\psi_j^2)$, $j=1,2$, satisfies to the system of differential equation:

$$\dot{\psi}_j^1(t)=-\nu tH^{-\gamma}e^{-\rho t}-\psi_j^1\big(\sigma\varepsilon H^{\varepsilon-1}sl-\delta\big)-\psi_j^2 rlg(s),\ j=1,2,$$

$$\dot{\psi}_j^2(t) = -i_j \psi_j^2 \,, \;\; j = 1, 2 \,;$$

3. transversability conditions:

$$\psi_2^1(T) = 0 \,,$$

$$\psi_2^2(T) = \Gamma K(T)^{-k} \,;$$

4. at τ which is the point of the switching surface penetration by the path function the conjugate vector-function and the Pontryagin function meet the jump condition

$$\psi_1(\tau - 0) = \psi_2(\tau + 0) + \lambda \frac{\partial S(\tau, \bar{x}(\tau))}{\partial x} \,, \;\; S(\tau, \bar{x}(\tau)) = 0 \,,$$

$$\mathrm{H}_1(\tau - 0) = \mathrm{H}_2(\tau + 0) \,,$$

$$\lambda = \frac{(f_2(\tau, x(\tau), u(\tau + 0)) - f_1(\tau, x(\tau), u(\tau - 0)), \psi_2(\tau))}{\left(\dfrac{\partial S(\tau, x(\tau))}{\partial x}, f_1(\tau, x(\tau), u(\tau - 0)) \right)} \,,$$

where λ is the magnitude of jump in the point τ.

Thus we have a boundary value problem of the maximum principle of Pontryagin. To solve it we use the Lagrange method of multipliers based on the simplifying the original continuous problem of optimal control (30) – (34) to the discrete one.

To find optimal control and trajectories we used a numerical software algorithm based on the gradient projection. Figure 3 and Figure 4 present the results of solving the discontinuous problem of optimal control. On the diagrams the curves H1(t), K1(t), l1(t), c1(t), s1(t) demonstrate the solutions of the discontinuous problem with the parameters $i_1 = 0{,}04$ and $i_2 = 0{,}06$, and the curves H0(t), K0(t), l0(t), c0(t), s0(t) are the solutions for the smooth right-hand problem with $i_1 = i_2 = i = 0{,}04$.

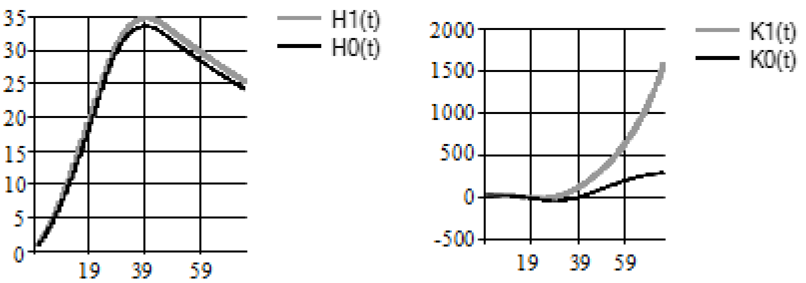

Fig. 3. Temporal changes of human and financial capital in smooth and discontinuous cases

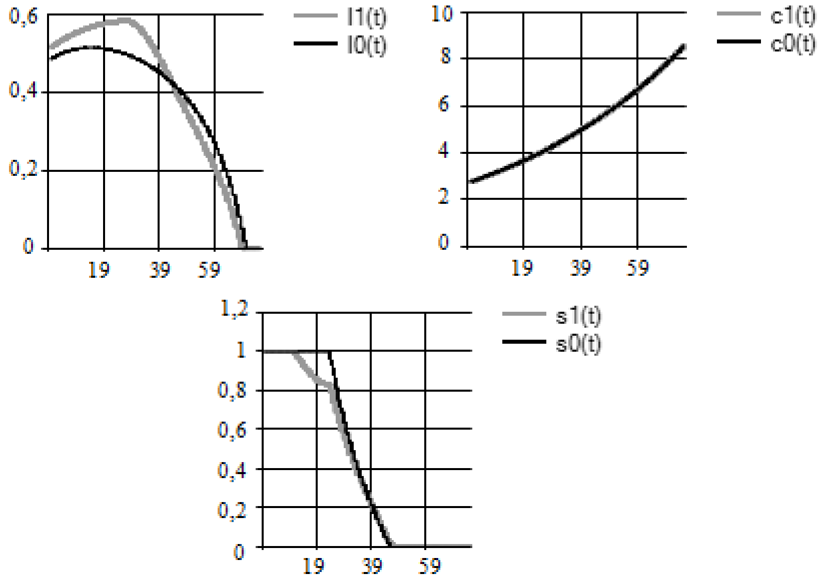

Fig. 4. Dependency of optimal control on time in smooth and discontinuous cases

Table 2 presents some numerical results like the final time for education (t_1), the final time for professional training (t_2), retirement age (t_3), left boundary of time with negative capital (t_4), the right boundary of time with negative capital (t_5), the age with maximum employment time ($t_{l\max}$) for smooth ($i_1 = 0{,}04$) and discontinuous ($i_2 = 0{,}06$) cases.

Table 2. Values of some numerical characteristics

	t_1	t_2	t_3	t_4	t_5	$t_{l\max}$
$i_1 = 0{,}04$	23,42	45,5	69,5	17,79	37,38	14,05
$i_2 = 0{,}06$	10,5	45,5	68,05	24,11	24,15	26,06

The results received after the solution of the discontinuous problem for optimal control prove the increase of “employment time” parameter in comparison with the results of the smooth problem and the maximum of this parameter is at the age of 26. Human capital demonstrates a slight increase. Professional education enters the educational experience on earlier stages. Because of the capital cost factor increase, the period with negative capital is reduced considerably. The increase of the capital cost reduces considerably the period with negative capital (Figure 5) and increases the growth rate of financial capital curve. The consumption is constant.

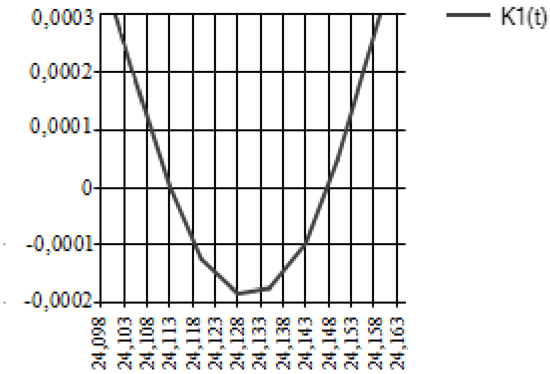

Fig. 5. Dynamics of capital changes close by zero

In conclusion we can state that changes in capital cost factor happen through the whole individual's life. These changes when caused by the discontinuity of dynamic restrictions in optimal control problem influence deeply in the allocation of human capital and the time period for professional training in education acquisition.

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