Parallel Partitioning Without Branching of Inner Boundaries for Arbitrary Domain*

Ilyas Kadyrov[1,2], Sergey Kopysov[1,2], and Alexander Novikov[1]

[1] Udmurt State University, Izhevsk, Russia
slasheeck@gmail.com
[2] Institute of Mechanics, Udmurt Federal Research Center UB RAS, Izhevsk, Russia

Abstract. In this paper we consider an approach to the parallel partitioning of a arbitrary domain into connected subdomains without branching of internal boundaries. The Reeb graph is a simplified representation of the topology of the required domain. Algorithms are shown in this paper is a modification of the presented earlier algorithm of the formation of the plane Reeb graph. The algorithm for constructing the volume Reeb graph allows us to determine the internal topology of the domain. A new algorithm of the formation of the partitioning into subdomains without branching of internal boundaries is presented.

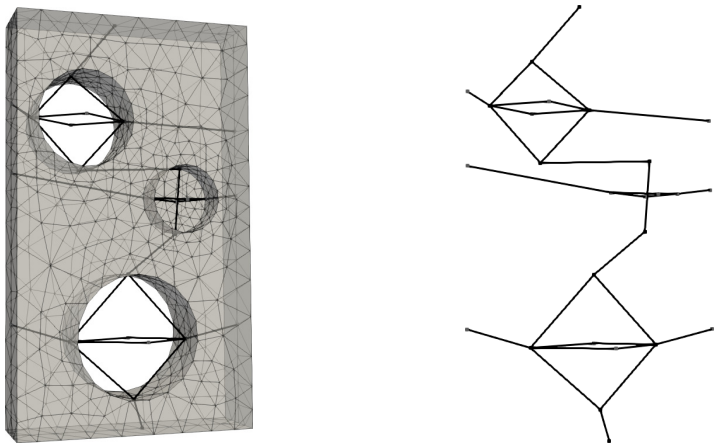
Keywords: Volume Reeb graph · Reeb graph · Unstructured mesh · Multiply connected domain · Topology definition · Mesh decomposition without branching

1 Reeb Graph Constructed by Surface Triangulation

A Reeb graph is a topological data structure which is able to capture the topology of shapes. Considering a scalar-valued function defined on the domain of interest, the level sets of this function may consist of several components, which evolve while sweeping through the function values. The Reeb graph encodes the evolution of these level set components, which are sometimes referred to as contours in the literature, by collapsing each contour to a point. Normal Reeb graph is a 2-dimentional graph, based and constructed on surface mesh. Volume Reeb graph is an array of subgraphs, that perform 3-dimentional graph.

Define a tree of connections between the critical nodes v^c and construct the Reeb graph $R(V, E)$ [1] with edges $e(v_i, v_j) \in E$ for the surface triangulation T in $\mathbb{R}^3$, consisting of 2-simplexes (Algorithm 1). We will use the Morse level function f to determine the critical nodes v^c and introduce the parameter $c(v^c)$ corresponding to the number of edges outgoing from it and taking the value: 1 - one node ; 2 - two nodes; 3 - branch; 0 is the auxiliary nodes v^a, which with the nearest critical v^c forms curvilinear arcs between the critical nodes in the branching case [2].

* This work is supported by the Russian Foundation for Basic Research (projects: 17-01-00402-a, 16-01-00129-a).



(a) (b)

Fig. 1. The mesh and volume Reeb graph: (a) mesh (b) volume Reeb graph with critical and auxilary nodes

---

Algorithm 1: Formation of the Reeb graph

Data: T — triangulation, consisting of 2-simplexes $\sigma \in T$;
f —level Morse function; n_t — number of available threads
Result: Reeb graph $R_i(V, E)$

while $m < |V(T)|$ **do**
  Partition nodes, according to coordinate $z(v)$, where $k = 1...n_t$
  Computing values of Morse function $f(v_k, m)$, for nodes in thread k
  $m \leftarrow m + 1$
Form layers of Morse function f
while $\left(\sum_{j=1}^{m} |LS_j(\sigma)| < |T|\right)$ **do**
  Each layer we will distribute between available threads $k = 0..n_t$
  forall v_k **do**
    $LS_m(\sigma) = \sum LS_m^k(\sigma) = \{\sigma \in Cl(LS(\alpha)) : \forall v \preceq \sigma, f(v_k) \geqslant \alpha\}$
  $m = m + 1$
foreach $LS(\alpha_m)$ **do**
  Define 2-simplexes σ, containing v_i **foreach** v_i **do**
    $St(v_i) = \{\sigma \in T | v_i \preceq \sigma\}$ link «star»
    foreach $v_h \in St$ **do**
      Define link of 0-simplexes v_h
      if $(Lk^+(v_h), Lk^-(v_h), Lk^{\pm}(v_h), Lk^{\mp}(v_h))$ **then**
        $v_h^c = v_h$
        $L(v_h^c) = l$, где $l = \{1, 2, 3, 4, 5\}$— link type of 0-simplexes

---

Find auxiliary nodes
foreach v_i^c **do**
  if $L(v_i^c) = 2 \wedge L(v_{i+1}^c) = 3$ **then**
    Find nodes v_i^a, v_{i+1}^a on surface line, between v_i^c and v_{i+1}^c.
    $v_{i+1}^c = v_i^a$ и $v_{i+2}^c = v_{i+1}^a$, $v^a \subseteq v^c$, $L(v_{i+1}^c) = 0 \wedge L(v_{i+1}^c) = 0$.
    Renumbering nodes and suppose that $i \to i + 2$
Form and compile Reeb graph $R_i(V, E)$

2 The Volume Reeb Graph

Define a tree of connections between the critical nodes v^c and construct the Reeb graph $R(V, E)$ with the edges $e(v_i, v_j) \in E$ for the triangulation T in $\mathbb{R}^3$, consisting of 3-simplexes [3].

We will use the Morse level function f of different directions in some coordinate planes. To determine the critical nodes v^c and introduce the parameter $c(v^c)$ corresponding to the number of edges outgoing from it, and taking the value: 1 - one node; 2 - two nodes; 3 - branch; 0 is an auxiliary point v^a, which with the nearest critical nodes v^c form curvilinear arcs between the critical nodes in the branching case [4].

Note by 0 the simplex v^s, as the "auxiliary". In contrast to the previously implemented approach to the formation of the Reeb graph, this algorithm uses a volume triangulation of the domain T.

Algorithm 2: Computing volume Reeb graph

Data: T — triangulation of 3-simplexes $\sigma \in T$; n_t — number of available threads.
Result: Volume Reeb graph $R(V, E)$

Compute graph by Algorithm 1 on result - 2d vertical Reeb graph $R_0(V, E)$
while $m < |V(T)|$ **do**
  for $k = 1 \quad to \quad n_t$ **do**
    For each thread we assign a separate subgraph.
    for $i \leqslant n$ **do**
      if v^a — *auxilary* **then**
        Form layer $S_i = \{\forall v \subseteq T \,|\, v \preceq z(v^a)$ of 3-simplexes
        S_i — level layer for triangulation of 2-simplexes T
      foreach S_n **do**
        Compile Reeb graphs $R_{n+1}(V, E)$ from Algorithm 1,
        separately for each S_n
        Bypassing the Morse function f, one of the horizontal axes
  Compile volume Reeb graph $R(V, E)$, where $R_{0..n}(V, E) \subseteq R(V, E)$
  $m = m + 1$

The volume Reeb graph $R(V, E)$ is represented as a set of subgraphs that are constructed from the mesh (or cells) layers in different directions of traversal of the Morse function.

The general graph has the notation $R_0(V, E)$, the remaining subgraphs will have the notation $R_{n+1}(V, E)$, where n_t is the number of inner layers.

3 Domain Partitioning by Volume Reeb Graph

Segmentation of subdomains into smaller fragments, which are blocks that represent split pants-decomposition. After obtaining the volume Reeb graph it is necessary to cut the domain T along the auxiliary nodes of the graph. The obtained layers S, which represent some contour of 2 or 3 simplexes, have some upper bound.

For each layer, the Morse function f with a different direction in the coordinate system is applied.

Algorithm 3: Partitioning of triangulation T on basis of Reeb graph

Data: T — triangulation of 3-simplexes $\sigma \in T$; $R(V, E)$ — Reeb graph
Result: Set of layers $\{LS_m(\sigma)\}$

We define auxiliary vertices $v^a = \{v^a \subseteq v^c \,|\, c(v^c) \equiv 0\}$
while $v_m^c < |V(R_0)|$ **do**
  Distribute each layer for auxilary nodes for available threads $k = 0..n_t$
  Looking for the nearest auxiliary nodes v_m^c for holes
  Abstractly draw the cutting planes with respect to 4 v^c along two axes
  Define the layers $LS_m^k(\sigma) = \sigma^- \in LS_m^k(\sigma), \sigma \notin LS_m^k(\sigma) \bigcup LS_{m-1}^k(\sigma)$
  $m \leftarrow m + 1$

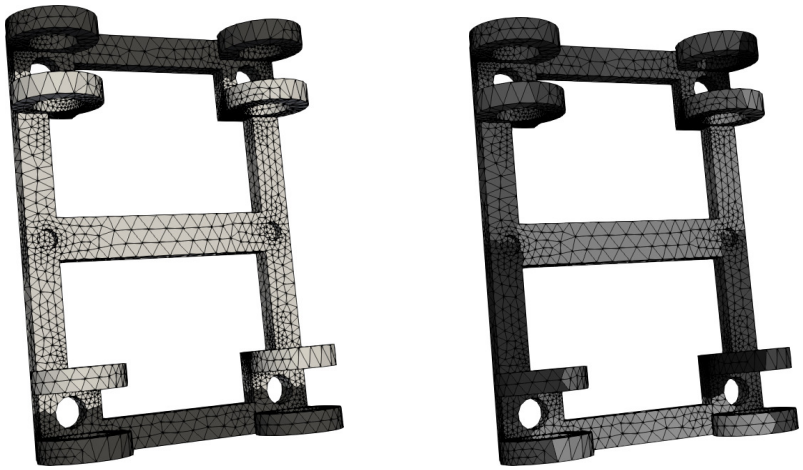
The resulting Reeb graph is associated with the main graph obtained in Algorithm 1 and is positioned relative to the auxiliary nodes[6]. Thus we obtain a volume connected graph which characterize the complete topology of the given domain.

An example of how the algorithm works is shown in the figure 2. On the figure 2 (a) presented partition of "frame" without branching of inner boundaries with only 1 secant axis. This decomposition cannot produce many subdomains without branching. On the figure 2 (b) decomposition produced in two axis on a large number of subdomains.

The topology of this domain and the graph characterizing it allow us to divide the domain into minimal independent parts.[5] The partitioning algorithm uses the Reeb graph for each layer, which makes it possible to implement a large number of algorithms.

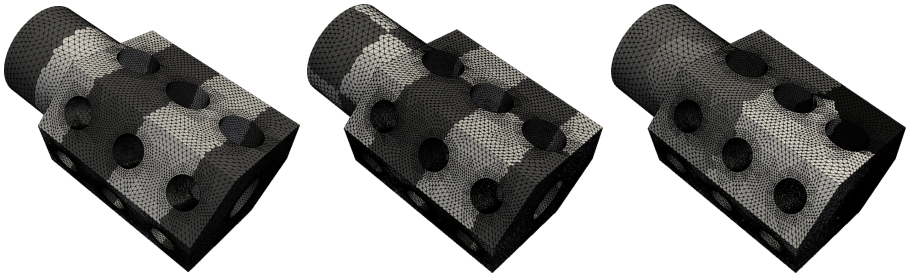
4 Numerical Experiments

Computational experiments were carried out on three-dimensional unstructured meshes for multiply connected regions, differing in the topological type, surface geometry and the details of its description by the mesh itself.



(a) (b)

Fig. 2. Partition on basis of Volume Reeb graph: (a) without branching of inner boundaries; (b) with branching



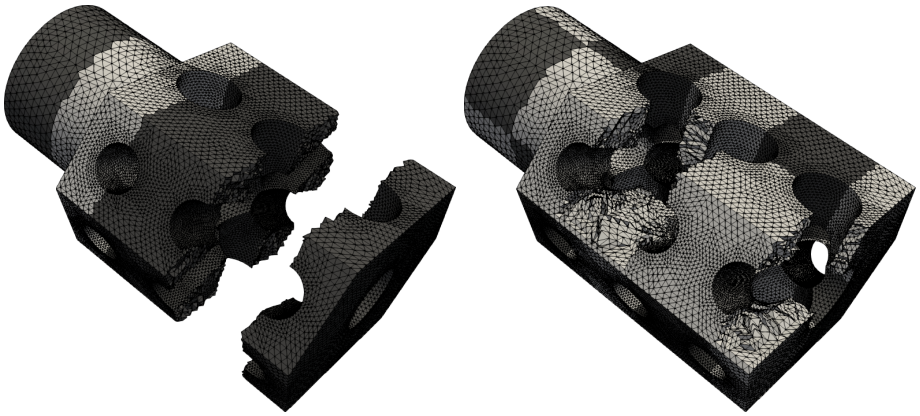
(a) (b) (c)

Fig. 3. The partitions: (a) without branching of inner boundaries; (b) partition with branching to 10 subdomains and (c) partition, made by METIS algorithm

Figure 3 presents us result of decomposition by volume Reeb graph and algorithm METIS. Decomposition produced by algorithm of volume Reeb graph on figure 3 (a), (b) uses topology of arbitary domain. Decomposition by the METIS algorithm partitions the domain based on cells connectivity, but does not limit the number of adjacent subdomains.

On figure 4 was hidden some subdomains to present the difference between (a) partition without branching of inner boundaries and (b) with branching.

At the moment, by parallelizing the main cycles and passes through 2 and 3 - simplexes of triangulation, the following estimates of computations are achieved. To measure estimate speed of algorithms in this paper we will use "speedup" $S = \frac{T_s}{T_{n_t}}$. The speedup is defined as the ratio of the serial runtime of the best



(a) (b)

Fig. 4. Hidden layers in partition of mesh: (a) without branching of inner boundaries; (b) with branching

sequential algorithm for solving a problem T_s to the time taken by the parallel algorithm time T_{n_t} to solve the same problem on n_t processors, cores or threads.

Table 1. Speedup by parallelizing of partitioning with Reeb graph

Threads	623552 nodes	1123102 nodes	7620096 nodes
Intel Xeon E5-1650 Six-core 12-threads 16 GB RAM			
T_s, sec	4103	6591	37532
4 (2 cores)	2,78	2,84	2,90
6	3,21	3,25	3,29
12	5,28	5,32	5,35
Intel i5-4460 Quad-core 4-threads 8 GB RAM			
T_s, s	3705	5772	33718
2	1,78	1,81	1,82
4	3,69	3,65	3,51

The results of parallel algorithm speedup shown in Table 1 was computed only on main parts of algorithm. The main parts of algorithm is compute of volume Reeb graph and partition on subdomains. We didn't measured parts of reading and writing mesh from external memory.

Source codes produced in C++ and parallel part made by using OpenMP. The algorithms considered in this paper make it possible to determine the features of the topology of various domains. The parallel version of algorithm was evaluated in $O(K \log^2 K/n)$, where n is the number of available threads and K is a number of 0-simplexes of mesh. The estimation of the computational costs of the partitioning algorithm by the Reeb graph is $O(L \log^2 L)$, where L is a sum

of all simplexes of mesh. Algorithms showed sufficiently accurate results, which indicates the reliability of their application.

The considered partitions, due to the elimination of branching of inner boundaries, reduce the number and simplify the structure of exchanges between computational processes in parallel algorithms of mesh methods.

The layer-by-layer ordering of mesh cells in the subdomains of the unstructured mesh excludes conflicts in computing in the shared memory of computational systems with a hybrid architecture.

Partitioning into a given number of subdomains can be provided by recursively dividing the obtained subdomains of an unstructured mesh and combining the layers of cells that provide independent access to the data.

References

1. Korneev V., Langer U. 2004 Encyclopedia of Computational Mechanics pp 617-647
2. Kadyrov I. R., Kopysov S. P., Novikov A. K. 2018 Partitioning of Triangulated Multiply Connected Domain into Subdomains Without Branching of Inner Boudaries (Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki) 160 [In print]
3. Kopysov S. P., Kuzmin I. M., Nedozhogin N. S., Novikov A. K. 2012 Parallel Algorithms for Constructing and Solving the Schur Complement on Graphics Accelerators (Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki) 154 (3) pp 202-215.
4. Martynenko S. I. 2008 Formalization of Computations at Numerical Solution of Boundary Value Problems (Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki) 150 (1) pp 76-90
5. Postnikov M. M. 1971 Introduction in Morse Theory (Moscow: Nauka) p 568
6. Zimovnov A. V., Mestetskiy L. M. 2016 On algorithm of curve-skeleton extraction for 3D model based on planar projections (Vestn. TvGU Seriya: Prikladnaya matematika) (3) pp 67-83
7. Hajij M., Dey T., Li X. 2016 Segmenting a surface mesh into pants using Morse theory (Graphical Models) 88 pp 12-21
8. Mestetskiy L. M. 2009 Continuous morphology of binary images: figures, skeletons and circulars (Moscow: Fizmatlit) p 288
9. Harvey W., Wang Y., Wenger R. 2010 A Randomized O. (M log M) Time Algorithm for Computing Reeb Graphs of Arbitary Simplical Complexes (USA: SoCG '10) p 267-276
10. Novikov A. K., Piminova N. K., Kopysov S. P., Sagdeeva Y. A. 2016 Layer-by-layer ordering in parallel finite composition on shared-memory multiprocessors (IOP Conf. Ser.: Mater. Sci. Eng.) 158 (1)