Regularization of an Inverse Problem by Controlling the Stiffness of the Graphs of Approximate Solutions * **

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Abstract. In this paper, an one-dimensional heat conductivity equation is considered. Such an equation describes, for example, a wall which exhibits temperature changes across the thickness, whereas the temperature remains constant along the in-plane directions. The problem of recovering the unknown temperature at the left end point of the domain is studied. It is assumed that the temperature and the heat flux are measured at the right end point of the domain. Using the Laplace transform, the problem is reduced to an integral equation defining the unknown temperature at the left end point as function of time. An approximation of the integral equation yields a linear system defining the values of the unknown function. Additionally, the graph of the unknown function is considered as a sequence of segments or overlapping quadratic or cubic parabolas, and the condition of common tangents at common points of neighboring parabolas is imposed. The resulting overdetermined system is solved using the least square method whose fitting function consists of two parts: a residual responsible for satisfying the integral equation and a term responsible for the matching of the segments/parabolas. The last term is multiplied by a regularization parameter that defines the stiffness of the solution graph. Appropriate values of the regularization parameter are being chosen as local minimizers of a discrepancy. Numerical experiments show that one of such values provides the best choice. Numerical simulations exhibit a very exact reconstruction of solutions even in the case of large measurement errors (up to 10%).

Keywords: Heat transfer · Inverse problems · Cauchy problems · Laplace transform · Regularization.

* The paper was carried out in the framework of the grant N0. 4917/B/T02/2010/39 funded by the Ministry of Highschool Education (Poland).

1 Introduction

Inverse problems, in contrast to stationary and non-stationary direct boundary value problems, are characterized by unknown boundary conditions on unreachable parts of boundaries. Such a situation is typical when studying the heat transfer in engineering objects that have complicated geometries with holes. The problem of cooling of a gas turbine casing can be mentioned as an example in this connection (see Fig. 1). Measurements give the temperature and the heat flux density on the outer boundary of a casing, whereas the temperature on channel walls of the casing should be recovered, see e.g. [1, 5]. Thus, the missing information about heat conditions in the unreachable part of the casing is compensated by a redundant condition, e.g. accounting for the heat flux on the outer casing boundary. Such problems named after Cauchy have been first considered by Hadamard who has observed that their solutions do not depend continuously on the given boundary data. Therefore, inverse problems belong to ill-conditioned ones in the Hadamard sense, [2], which means that small disturbances of boundary data cause large errors and oscillations in solutions. Such instabilities can destroy numerical procedures since the measured values of the temperature and the heat flux are always affected by errors. For example, the presence of temperature sensors can essentially disturb the measurement of the heat flux. To suppress instabilities typical for inverse problems, Tikhonov's regularization techniques based on the minimization of fitting functionals are used. The idea of Tikhonov's method consists in including a regularization term into the fitting functional. This provides the uniqueness and physically stipulated regularity of solutions.

Cauchy problems are being intensively studied because of their practical significance, see works [3, 4, 6, 8–11] for an exemplarily overview of stable approximation methods for solving ill-posed inverse problems. In paper [3], the problem is reduced to a linear second-kind integral Volterra equation which admits a unique solution. The method of fundamental solutions is used in paper [9] for solving a steady-state Cauchy problem. In papers [4] and [6], a finite difference method supplemented by Fourier transform techniques is applied. Legendre polynomials are used in paper [11] for the solution of an one-dimensional Cauchy problem. Wavelet-Galerkin method supplemented by the Fourier transform is utilized in paper [10]. The questions of uniqueness of solutions of Cauchy problems are considered in paper [8].

The purpose of the paper presented is to propose a stable method for solving an one-dimensional Cauchy problem related to reconstructing the temperature of one surface of a wall using measurements of the temperature and the heat flux on the other wall surface. The main features of the method consist in the reduction of the problem to an integral equation using the Laplace transform, approximation of solutions by sequences of segments or overlapping quadratic or cubic parabolas, and imposing the condition of common tangents at common points of neighboring segments/parabolas, which controls the rigidity of the graphs of approximate solutions. Numerical simulations show a very exact reconstruction of solutions even in the case of large measurement errors (up to 10%).

2 Model Equation

The governing equation and the initial and boundary conditions are the following:

$$\rho c \cdot \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right), \quad x \in (0, \delta), \quad t > 0, \tag{1}$$

$$T(x, 0) = T_0(x), \tag{2}$$

$$T(\delta, t) = H(t), \tag{3}$$

$$-\lambda \frac{\partial T}{\partial x}(\delta, t) = Q(t), \tag{4}$$

$$T(0, t) = F(t). \tag{5}$$

Here, ρ denotes the density, c the relative heat, and λ is the heat conductivity coefficient. The temperature $F(t)$ is unknown.

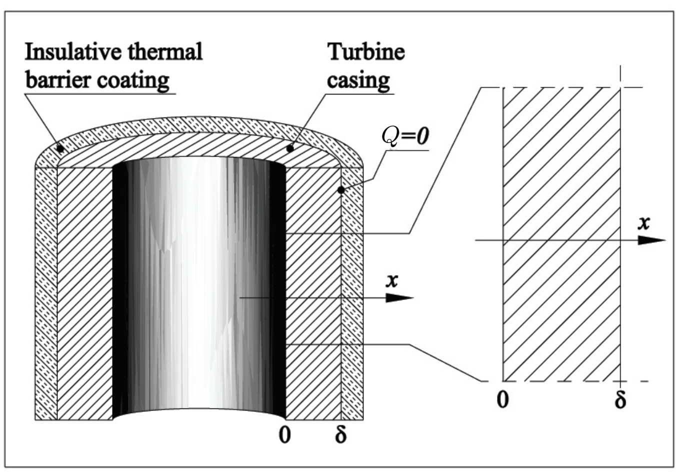

Fig. 1. Fragment of a gas turbine casing with a one-dimensional wall permitting the heat transfer across the thickness only (left); calculation area (right).

Obviously, the problem described by (1)–(4) is a Cauchy problem because the boundary conditions (3) and (4) are imposed at the same point $x = \delta$.

For the next considerations, it is convenient to introduce non-dimensional variables

$$\vartheta = \frac{T}{T_{\max}}, \quad \xi = \frac{x}{\delta}, \quad \tau = \frac{\lambda}{\rho c} \cdot \frac{t}{\delta^2}, \quad \text{where} \quad T_{\max} = \max_{\substack{x \in (0,\delta) \\ t \geq 0}} T(x,t), \tag{6}$$

to obtain the following non-dimensional formulation of the problem:

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2}, \quad \xi \in (0,1), \quad \tau > 0, \tag{7}$$

$$\vartheta(\xi, 0) = \vartheta_0(\xi) := T_0(x)/T_{\max}, \quad \xi \in (0,1), \tag{8}$$

$$\vartheta\,(1,\tau) = h\,(\tau) := H(t)/T_{\max}, \quad \tau > 0, \tag{9}$$

$$-\frac{\partial \vartheta}{\partial \xi}\,(1,\tau) = q\,(\tau) := \frac{\delta}{\lambda \cdot T_{\max}} \cdot Q(t), \quad \tau > 0. \tag{10}$$

Finally, the unknown boundary temperature at $\xi = 0$ reads

$$\vartheta\,(0,\tau) = \chi\,(\tau) := F(t)/T_{\max}, \quad \tau > 0. \tag{11}$$

3 Analytical Solution

3.1 Laplace Transform

In view of linearity of equations (7)–(10), the Laplace transform can be applied. Denote

$$L\vartheta\,(\xi,\tau) = \bar{\vartheta}\,(\xi,s) := \int_0^\infty \vartheta\,(\xi,\tau) \cdot e^{-s\tau} d\tau \tag{12}$$

and observe that the system of equations (7)–(10) is transformed to the form

$$s \cdot \bar{\vartheta}\,(\xi,s) - \vartheta\,(\xi,0) = \frac{d^2 \bar{\vartheta}}{d\xi^2}, \tag{13}$$

$$\bar{\vartheta}\,(\xi,0) = \bar{\vartheta}_0\,(\xi)\,, \tag{14}$$

$$\bar{\vartheta}\,(1,s) = \bar{h}\,(s)\,, \tag{15}$$

$$-\frac{\partial \bar{\vartheta}}{\partial \xi}\,(1,s) = \bar{q}\,(s)\,. \tag{16}$$

The unknown boundary condition at $\xi = 0$ reads

$$\bar{\vartheta}\,(0,s) = \bar{\chi}\,(s)\,. \tag{17}$$

For simplicity, assume that $\vartheta_0(\xi) = \vartheta_0 = const$. Then the solution of the direct problem given by (13), (14), (16), and (17) has the form

$$\begin{aligned}\bar{\vartheta}(\xi, s) =& \bar{\chi}(s) \cdot \frac{\cosh\sqrt{s}\,(1-\xi)}{\cosh\sqrt{s}} - \\ & \bar{q}(s) \cdot \frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}} + \frac{\vartheta_0}{s}\cdot\left(1 - \frac{\cosh\sqrt{s}\,(1-\xi)}{\cosh\sqrt{s}}\right).\end{aligned} \tag{18}$$

Rewrite the previous equation as

$$\begin{aligned}\bar{\vartheta}(\xi, s) =& s\cdot\bar{\chi}(s) \cdot \frac{\cosh\sqrt{s}\,(1-\xi)}{s\cdot\cosh\sqrt{s}} - \\ & s\cdot\bar{q}(s) \cdot \frac{1}{s}\cdot\frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}} + \vartheta_0\left(\frac{1}{s} - \frac{\cosh\sqrt{s}\,(1-\xi)}{s\cosh\sqrt{s}}\right)\end{aligned} \tag{19}$$

and observe that the poles of the right-hand side of (19) are given by the relations

$$s = 0 \quad \text{and} \quad \cosh(\sqrt{s}) = 0.$$

Setting $\sqrt{s} = i\mu$ and observing that $\cosh(i\mu) = \cos(\mu)$ yield the following roots of the equation $\cosh(\sqrt{s}) = 0$:

$$s_n = -\mu_n^2, \quad \text{where} \quad \mu_n = (2n-1)\cdot\frac{\pi}{2},\ n = 1, 2, \ldots\ .$$

3.2 Inverse Laplace Transform

Let $L^{-1}f$ and $\mathrm{Res}_x f$ denote the inverse Laplace transform of f and the residue of f at x, respectively. The following auxiliary calculations are true:

$$\begin{aligned}& L^{-1}\left[\frac{\cosh\sqrt{s}\,(1-\xi)}{s\cdot\cosh\sqrt{s}}\right] = \\ & \operatorname*{Res}_{s=0}\frac{\cosh\sqrt{s}\,(1-\xi)}{s\cdot\cosh\sqrt{s}} + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n}\frac{\cosh\sqrt{s}\,(1-\xi)}{s\cdot\cosh\sqrt{s}}\cdot e^{s\cdot\tau} = \\ & 1 + \sum_{n=1}^{\infty}\lim_{s\to s_n}\frac{(s-s_n)\cdot\cosh\sqrt{s}\,(1-\xi)}{s\cdot\cosh\sqrt{s}}\cdot e^{s\tau} = \\ & 1 - 2\sum_{n=1}^{\infty}\frac{\cos\mu_n(1-\xi)}{\mu_n\cdot\sin\mu_n}\cdot e^{-\mu_n^2\cdot\tau} = \\ & 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\mu_n\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau},\end{aligned} \tag{20}$$

$$L^{-1}\left[\frac{1}{s}\cdot\frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}}\right] = \xi - 2\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{\sin\mu_n\xi}{\mu_n^2}\cdot e^{-\mu_n^2\tau}, \tag{21}$$

$$L^{-1}\left[s\,\bar{q}\,(s)\right] = q'\,(\tau) + q_0\cdot\delta\,(\tau)\,, \quad L^{-1}\left[s\,\bar{\chi}\,(s)\right] = \chi'\,(\tau) + \chi_0\cdot\delta\,(\tau)\,, \\ \big(\text{cf. e.g. } L\left[q'\,(\tau)\right] = s\cdot\bar{q}\,(s) - q_0\big), \tag{22}$$

$$L^{-1}\left[\frac{1}{s}\right] = \eta(\tau), \tag{23}$$

where $\delta\,(\tau)$ is Dirac's delta function, $\eta(\tau)$ is the unit step function, $q_0 = q(0)$, and $\chi_0 = \chi(0)$.

Application of the above auxiliary calculations yields the following results:

$$\begin{gathered}
\vartheta\,(\xi,\tau) = L^{-1}\left[\bar{\vartheta}\,(\xi,s)\right] = \vartheta_0\left[\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\tau}\right] + \\
L^{-1}\left[s\,\bar{\chi}\,(s)\right] * \left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\tau}\right] - \\
L^{-1}\left[s\,\bar{q}\,(s)\right] * \left(\xi - \frac{8}{\pi^2}\sum_{n-1}^{\infty}(-1)^{n-1}\,\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{(2n-1)^2}\cdot e^{-\mu_n^2\tau}\right) = \\
= \vartheta_0\left[\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\tau}\right] + \left[\chi'\,(\tau) + \chi_0\cdot\delta\,(\tau)\right] * \eta\,(\tau) - \\
\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\tau}\cdot\int_0^{\tau}\left[\chi'\,(p) + \chi_0\cdot\delta\,(p)\right]\cdot e^{\mu_n^2 p}\cdot dp - \\
\left[q'\,(\tau) + q_0\cdot\delta\,(\tau)\right] * \eta\,(\tau)\cdot\xi\, + \\
\frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{(2n-1)^2}\cdot e^{-\mu_n^2\tau}\cdot\int_0^{\tau}\left[q'\,(p) + q_0\cdot\delta\,(p)\right]e^{\mu_n^2 p}\cdot dp = \\
= \vartheta_0\left[\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\tau}\right] + \chi\,(\tau)\cdot\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1}\right] + \\
2\cdot\sum_{n=1}^{\infty}\mu_n\cdot\sin\mu_n\xi\cdot e^{-\mu_n^2\tau}\cdot\int_0^{\tau}\chi\,(p)\cdot e^{\mu_n^2 p}\cdot dp - \\
\sum_{n=1}^{\infty}\sin\,(2n-1)\,\frac{\pi}{2}\cdot\xi\cdot e^{-\mu_n^2\tau}\cdot\int_0^{\tau}q\,(p)\cdot e^{\mu_n^2 p}\cdot dp.
\end{gathered} \tag{24}$$

Since

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\,(2n-1)\,\frac{\pi}{2}\xi}{2n-1} = 1$$

for $\xi > 0$, the square bracket following $\chi(\tau)$ vanishes.

4 Inverse Problem

Henceforth, the case where $q(\tau) \equiv 0$ and $\vartheta_0 = 0$ will be considered. Then equation (24) assumes the form

$$\vartheta(\xi,\tau) = \int_0^{\tau} \chi(p) \cdot \psi(\xi,\tau,p)\ dp, \quad \xi \in (0,1),\ \tau \geq 0, \tag{25}$$

where

$$\psi(\xi,\tau,p) = 2\sum_{n=1}^{\infty} \mu_n \cdot \sin \mu_n \xi \cdot e^{-\mu_n^2(\tau-p)}. \tag{26}$$

The boundary condition (9) and formula (25) yield the integral equation

$$\int_0^{\tau} \chi(p) \cdot \psi(1,\tau,p) \cdot dp = h(\tau) \tag{27}$$

for the determination of $\chi(\tau)$.

4.1 Approximation of the Integral Equation

Assume that the temperature $h(t)$ is measured with the sampling time $\Delta\tau$ so that the values $h_k = h(\tau_k),\ \tau_k = k \cdot \Delta\tau,\ k = 0,1,2,\ldots,$ are available. Then equation (27) assumes the form

$$\int_0^{\tau_k} \chi(p) \cdot \psi_k(p) \cdot dp = h_k, \quad \text{with} \quad \psi_k(p) = \psi(1,\tau_k,p). \tag{28}$$

Using the sampling $\chi_j = \chi(\tau_j),\ j = 0,\ldots,k$, and choosing a mixing coefficient $\Theta \in (0,1)$, we have

$$\int_0^{\tau_k} \chi(p) \cdot \psi_k(p) dp = \sum_{j=1}^{k} \int_{\tau_{j-1}}^{\tau_j} \chi(p) \cdot \psi_k(p)\, dp \approx$$

$$\sum_{j=1}^{k} \int_{\tau_{j-1}}^{\tau_j} \left[\Theta \cdot \chi_{j-1} + (1-\Theta)\chi_j\right] \cdot \psi_k(p)\, dp =$$

$$\sum_{j=1}^{k} \left[\Theta \cdot \chi_{j-1} \cdot r_{kj} + (1-\Theta) \cdot \chi_j \cdot r_{kj}\right] =$$

$$\Theta \cdot \chi_0 \cdot r_{k1} + (1-\Theta)\chi_k \cdot r_{kk} + \sum_{j=1}^{k} \chi_j \left[\Theta \cdot r_{kj+1} + (1-\Theta) \cdot r_{kj}\right] =: \sum_{j=0}^{k} \chi_j \psi_{kj}.$$

Here, for any $k \geq 1$,

$$
\begin{aligned}
& r_{kj} = \int\limits_{\tau_{j-1}}^{\tau_j} \psi_k(p)\, dp, \quad j = 1, \dots k, \\
& \psi_{kj} = \Theta \cdot r_{kj+1} + (1 - \Theta)\, r_{kj}, \quad j = 1, \dots k-1, \\
& \psi_{k0} = \Theta \cdot r_{k1}, \quad \psi_{kk} = (1 - \Theta)\, r_{kk}.
\end{aligned} \tag{29}
$$

Thus, the approximation of (28) can be written as:

$$
\sum_{j=0}^{k} \chi_j \psi_{kj} = h_k, \quad k = 1, \dots M,
$$

where M defines the time horizon. The matrix form of this system reads

$$
[\psi]\{\chi\} = \{h\}, \quad \dim[\psi] = M \times (M+1), \quad \dim\{h\} = M. \tag{30}
$$

Remark. Use (26) to explicitly obtain

$$
\begin{aligned}
r_{kj} = \int\limits_{\tau_{j-1}}^{\tau_j} \psi(1, \tau_k, p) dp = 2 \sum_{n=1}^{\infty} \mu_n \cdot \sin\mu_n \cdot \int\limits_{\tau_{j-1}}^{\tau_j} e^{-\mu_n^2(\tau_k - p)} dp = \\
= 2 \sum_{n=1}^{\infty} \frac{\sin\mu_n}{\mu_n} \cdot \Big(e^{-\mu_n^2 \Delta\tau \cdot (k-j)} - e^{-\mu_n^2 \Delta\tau \cdot (k-j+1)} \Big).
\end{aligned} \tag{31}
$$

Note that $r_{kj} = r_{k+l,j+l}$, and hence $\psi_{kj} = \psi_{k+l,j+l}$ for any $l > 0$, which saves the computation efforts.

4.2 A Known Particular Solution

In order to validate the solution of integral equation (27) via approximation (30), use a known particular solution of equation (7) with the initial value $\vartheta(\xi, 0) = 0$ and the boundary conditions

$$
\vartheta(0, \tau) = T_b \cdot \left(1 - e^{-\beta\tau}\right), \quad -\frac{\partial\vartheta}{\partial\xi}(1, \tau) - Bi \cdot \vartheta(1, \tau) = 0. \tag{32}
$$

Here, Bi is the Biot number.

The solution has the form

$$
\begin{aligned}
\vartheta(\xi, \tau) = & T_b \cdot \left(1 - \frac{Bi}{Bi+1} \cdot \xi\right) \left(1 - e^{-\beta\tau}\right) + \\
& 2T_b \cdot \beta \cdot e^{-\beta\tau} \cdot \sum_{n=1}^{\infty} w_n(\xi) \cdot \frac{1}{p_n^2 - \beta} - \\
& 2T_b \cdot \beta \cdot \sum_{n=1}^{\infty} w_n(\xi) - \frac{1}{p_n^2 - \beta} \cdot e^{-p_n^2 \tau},
\end{aligned} \tag{33}
$$

where

$$w_n(\xi) = -\frac{\sin p_n \xi}{p_n} \cdot \left(1 - \frac{Bi}{Bi^2 + Bi + p_n^2}\right),$$

and p_n are the roots of the equation

$$\tan p_n = -\frac{p_n}{Bi}, \quad n = 1, 2, \ldots, \quad \lim_{Bi \to 0} \left(p_n - \frac{\pi}{2}(2n-1)\right) = 0, \text{ uniformly in } n.$$

Thus, if $Bi = 0$, we can set $r(\tau) = \partial\vartheta(1,\tau)/\partial\xi = 0$, $h(\tau) = \vartheta(1,\tau)$, and $\chi(\tau) = T_b \cdot \left(1 - e^{-\beta\tau}\right)$. Therefore, the solution given by approximation (30) can be compared with $\chi(\tau)$. Computer experiments show that system (30) is numerically unstable (see Fig. 2). The instability occurs near to the time horizon $M \cdot \Delta$. Thus, the regularization of solutions of (30) is necessary.

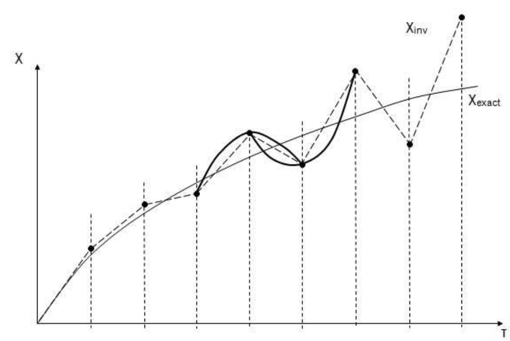

Fig. 2. Oscillations of a solution given by non-regularized system (30).

4.3 Regularization of solutions

Remember that approximate solutions are searched as grid functions assuming values χ_j at grid points τ_j, $j = 0, \ldots M$. The idea of the regularization is to introduce some rigidity to the graph of the approximation. This can be done by accounting for approximate first, second, third, or fourth derivatives.

Regularization with the 2nd derivative. Consider two adjacent intervals $[\tau_{i-1}, \tau_i]$ and $[\tau_i, \tau_{i+1}]$ (see Fig. 3) and denote the finite difference approximations of the left and right first derivatives at τ_i by $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$, respectively. Impose the condition

$$\delta_i = \tilde{\chi} - \tilde{\tilde{\chi}} = \frac{\chi_i - \chi_{i-1}}{h} - \frac{\chi_{i+1} - \chi_i}{h} = -\frac{1}{h}\left(\chi_{i+1} - 2\chi_i + \chi_{i-1}\right) \approx 0 \qquad (34)$$

that expresses a small jump of the approximate first derivative at τ_i.

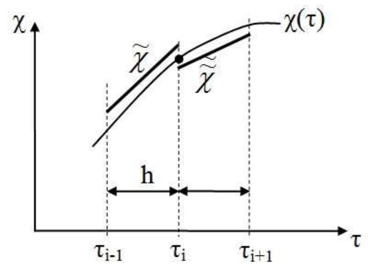

Fig. 3. Fitting of the approximate first derivatives.

Note that

$$\frac{\chi_{i+1} - 2\chi_i + \chi_{i-1}}{h^2} = \chi''(\tau_i) + 0(h), \quad i = 1, 2, \ldots, M-1,$$

and therefore

$$\sum_{i=1}^{M-1} \left(\frac{\chi_{i+1} - 2\chi_i + \chi_{i-1}}{h^2} \right)^2 \approx \int_0^t \left(\chi''(\tau) \right)^2 dt =: J_1.$$

The matrix corresponding to the relation (34) has the following form:

$$[w] = \begin{bmatrix} 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & \\ & & & \cdots & & & \\ & & & & 1 & -2 & 1 \end{bmatrix}, \quad \dim[w] = (M-2) \cdot M, \tag{35}$$

or

$$[w] = \begin{bmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & \\ & & & \cdots & & & \\ & & & & 1 & -2 & 1 \end{bmatrix}, \quad \dim[w] = (M-1) \cdot M,$$

if the value of $\chi'(0)$ is available.

Regularization with the 3rd derivative. Consider now three adjacent intervals $[\tau_{i-1}, \tau_i]$, $[\tau_i, \tau_{i+1}]$, and $[\tau_{i+1}, \tau_{i+2}]$ and denote now by $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ the finite difference approximations of the second derivative at τ_i and τ_{i+1}, respectively. Impose the condition

$$\begin{aligned} \delta_i = \tilde{\chi} - \tilde{\tilde{\chi}} = \frac{\chi_{i+1} - 2\chi_i + \chi_{i-1}}{h^2} - \frac{\chi_{i+2} - 2\chi_{i+1} + \chi_i}{h^2} = \\ \frac{1}{h^2} \left(\chi_{i-1} - 3\chi_i + 3\chi_{i+1} - \chi_{i+2} \right) \approx 0 \end{aligned} \tag{36}$$

that guarantees the closeness of the parabolas shown in Fig. 4 on the common interval $[\tau_i, \tau_{i+1}]$.

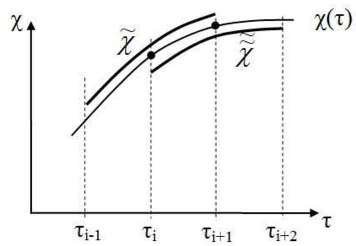

Fig. 4. Fitting of the approximate second derivatives.

Note that

$$\frac{\chi_{i-1} - 3\chi_i + 3\chi_{i+1} - \chi_{i+2}}{h^3} = -\chi^{'''}(\tau_i) + 0(h), \quad i = 1, 2, \ldots, M-2,$$

and therefore

$$\sum_{i=1}^{M-2} \left(\frac{\chi_{i-1} - 3\chi_i + 3\chi_{i+1} - \chi_{i+2}}{h^3} \right)^2 \approx \int_0^t \left(\chi^{'''}(\tau) \right)^2 dt =: J_2.$$

The matrix corresponding to the relation (36) has the following form:

$$[w] = \begin{bmatrix} 1 & -3 & 3 & -1 & & & \\ & 1 & -3 & 3 & -1 & & \\ & & \cdots & \cdots & \cdots & & \\ & & & 1 & -3 & 3 & -1 \end{bmatrix}, \quad \dim[w] = (M-2) \cdot M. \tag{37}$$

Regularization with the 4rth derivative. Consider fourth adjacent intervals formed by the points $\tau_{i-2}, \tau_{i-1}, \tau_i, \tau_{i+1}, \tau_{i+2}$, and two cubic parabolas corresponding to the points $\{\tau_{i-2}, \tau_{i-1}, \tau_i, \tau_{i+1}\}$ and $\{\tau_{i-1}, \tau_i, \tau_{i+1}, \tau_{i+2}\}$, respectively (see Fig. 5). Impose the condition

$$\chi_{i-2} - 4\chi_{i-1} + 6\chi_i - 4\chi_{i+1} + \chi_{i+2} \approx 0 \tag{38}$$

that guarantees the closeness of the cubic parabolas on the common intervals $[\tau_{i-1}, \tau_i]$ and $[\tau_i, \tau_{i+1}]$.

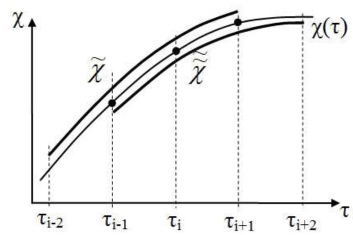

Fig. 5. Fitting of the approximate third derivatives.

It easily to see that

$$\frac{\chi_{i-2}-4\chi_{i-1}+6\chi_i-4\chi_{i+1}+\chi_{i+2}}{h^4}=\chi^{''''}(\tau_i)+0\,(h)\,,\quad i=2,3,\ldots,M-2,$$

and therefore

$$\sum_{i=2}^{M-2}\left(\frac{(\chi_{i-2}-4\chi_{i-1}+6\chi_i-4\chi_{i+1}+\chi_{i+2})}{h^4}\right)^2\approx\int_0^t\left(\chi^{''''}(\tau)\right)^2dt=:J_3.$$

The matrix corresponding to the relation (38) has the following form:

$$[w]=\begin{bmatrix}1&-4&6&-4&1&&&\\&1&-4&6&-4&1&&\\&&\cdots&\cdots&\cdots&\cdots&&\\&&&1&-4&6&-4&1\end{bmatrix},\quad \dim[w]=(M-4)\cdot M. \tag{39}$$

When solving equation (30), conditions (34), (36), and (38) can be accounted for by minimizing the following functional (the upper index ς denotes the noise level in data measured with error):

$$J\left(\{\chi\}\right)=\left\|[\psi]\{\chi\}-\{h\}^{\varsigma}\right\|^2+\alpha^2\left\|[w]\{\chi\}\right\|^2,$$

which can be interpreted as the application of minimum square method to the system

$$\begin{bmatrix}[\psi]\\ \alpha\,[w]\end{bmatrix}\{\chi\}=\begin{Bmatrix}\{h\}^{\varsigma}\\ \{0\}\end{Bmatrix}. \tag{40}$$

The optimality condition (the zero first variation of the functional J) reads

$$\frac{1}{2}\delta J\left(\{\chi\}\right)=[\psi]^T[\psi]\{\chi\}+\alpha^2[w]^T[w]\{\chi\}-[\psi]^T\{h\}^{\varsigma}=0,$$

and therefore

$$\left([\psi]^T[\psi]+\alpha^2[w]^T[w]\right)\{\chi\}=[\psi]^T\{h\}^{\varsigma},\ \ \operatorname{rank}\left([\psi]^T[\psi]+\alpha^2[w]^T[w]\right)=M.$$

Finally

$$\{\chi\}_\alpha^\varsigma = \Big([\psi]^T [\psi] + \alpha^2 [w]^T [w] \Big)^{-1} [\psi]^T \{h\}^\varsigma. \tag{41}$$

Let $[\psi]^+ := \lim_{\epsilon\to 0}([\psi]^T[\psi]+\epsilon I)^{-1}[\psi]^T$ be the Moore-Penrose-Inverse matrix, and $\{\chi\}^+ := [\psi]^+\{h\}$ a unique Moore-Penrose solution of (30). Estimate the difference

$$\begin{aligned}
&\{\chi\}^+ - \{\chi\}_\alpha^\varsigma = [\psi]^+\{h\} - \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\{h\}^\varsigma = \\
&[\psi]^+\{h\} - \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\{h\} + \\
&\left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T \cdot (\{h\} - \{h\}^\varsigma) = \\
&\Big([\psi]^+ - \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\Big) \cdot \{h\} + \\
&\left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1} \cdot (\{h\} - \{h\}^\varsigma)\,.
\end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned}
&\left\|\{\chi\}^+ - \{\chi\}_\alpha^\varsigma\right\|_2 \le \left\|[\psi]^+ - \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\right\|_2 \cdot \|\{h\}\|_2 + \\
&\left\|\left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\right\|_2 \cdot \|\{h\} - \{h\}^\varsigma\|_2 \le \\
&\left\|[\psi]^+ - \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\right\|_2 \cdot \|\{h\}\|_2 + \\
&\left\|\left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T\right\|_2 \cdot \varsigma = \\
&\left\|[\psi]^+ - E(\alpha)\right\|_2 \cdot \|\{h\}\|_2 + \|E(\alpha)\|_2 \cdot \varsigma, \quad \text{where} \\
&E(\alpha) = \left([\psi]^T[\psi] + \alpha^2[w]^T[w]\right)^{-1}[\psi]^T, \quad \text{rank} \;\; E = M.
\end{aligned} \tag{43}$$

In numerical computations, the distance $\|\{\chi\}^+ - \{\chi\}_\alpha^\varsigma\|_2$ will be used for finding optimal values of $\alpha(\varsigma)$ for given noise levels ς. Moreover, if an exact stable solution $\{\chi^0\}$ of (30) would be known, say $\{\chi^0\} = \{\chi\}^+$, its rigidity can be compared with that of the solution searched. This motivates the following modification of the objective functional:

$$J\left(\{\chi\},\{\chi^0\}\right) = \|[\psi]\{\chi\} - \{h\}\|^2 + \alpha^2\left\|[w]\left(\{\chi\} - \{\chi^0\}\right)\right\|^2 \tag{44}$$

which corresponds to the system

$$\begin{bmatrix} [\psi] \\ \alpha\,[w] \end{bmatrix}\{\chi\} = \begin{Bmatrix} \{h\} \\ \alpha\,[w]\{\chi^0\} \end{Bmatrix} = \begin{Bmatrix} \{h\} \\ \{0\} \end{Bmatrix} + \begin{bmatrix} [0] \\ \alpha\,[w] \end{bmatrix}\{\chi^0\}\,. \tag{45}$$

Figures 6–9 show simulation results.

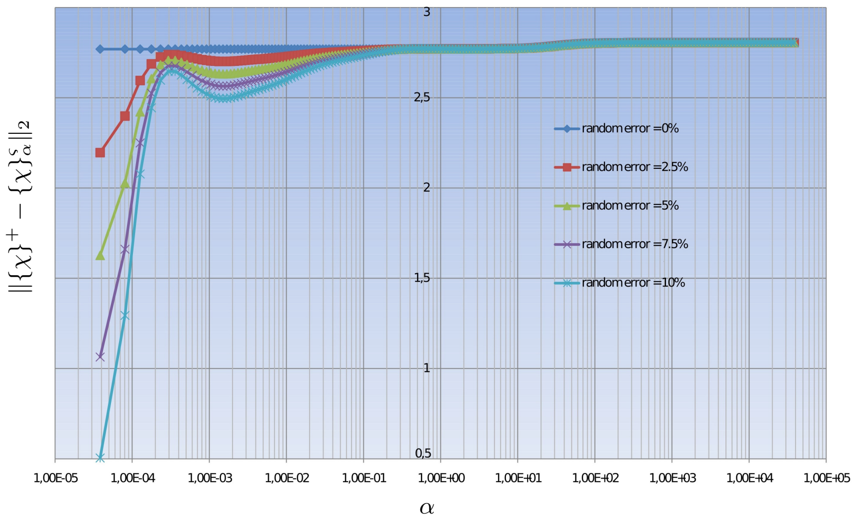

Fig. 6. Dependency of the distance between $\{\chi\}^+$ and $\{\chi\}_\alpha^\varsigma$ on α at fixed error levels ς. Appropriate values of α are local minimizers of this function. The case of matrix (35) is considered.

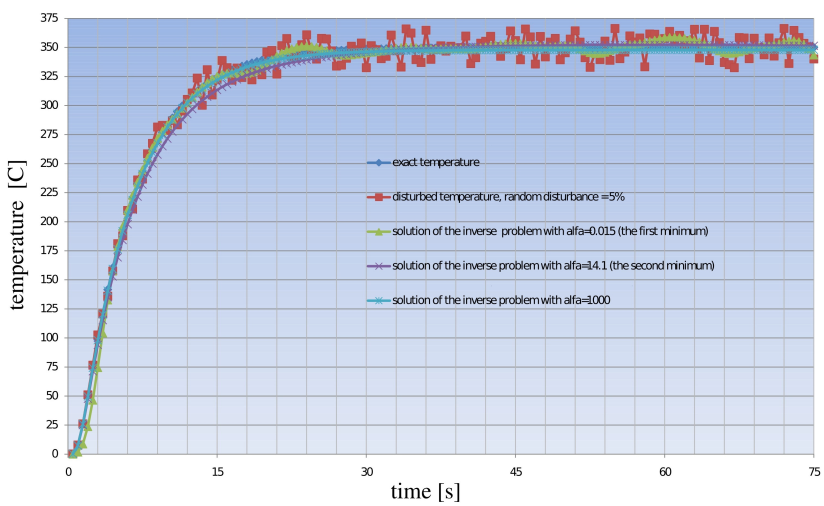

Fig. 7. Comparison of the exact temperature at $x = 1$ with the temperature reconstructed from the inverse problem for several appropriate values (local minimizers, see Fig. 6) of the regularization parameter α. The random noise level ς equals 5.0%. The case of matrix (35) is considered.

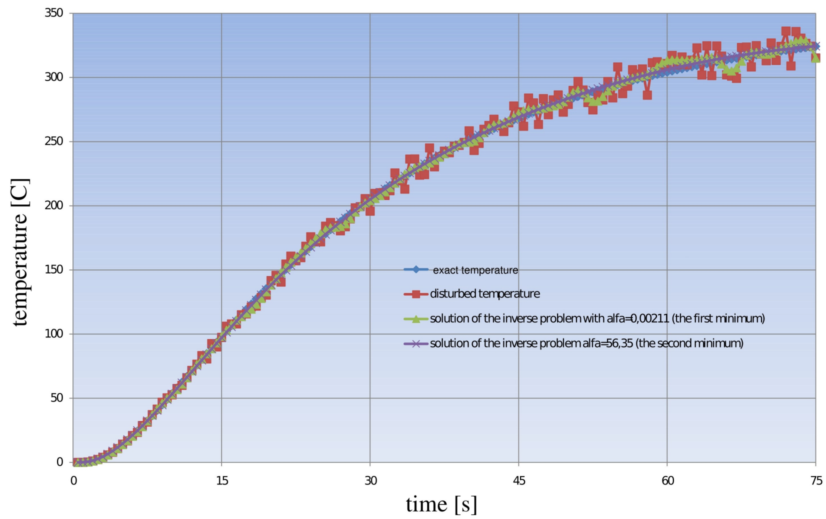

Fig. 8. Comparison of the exact temperature at $x = 0$ with the temperature obtained from the inverse problem for several appropriate values (local minimizers, compare with Fig. 6) of the regularization parameter α. The random noise level ς equals 5.0%. The case of matrix (37) is considered.

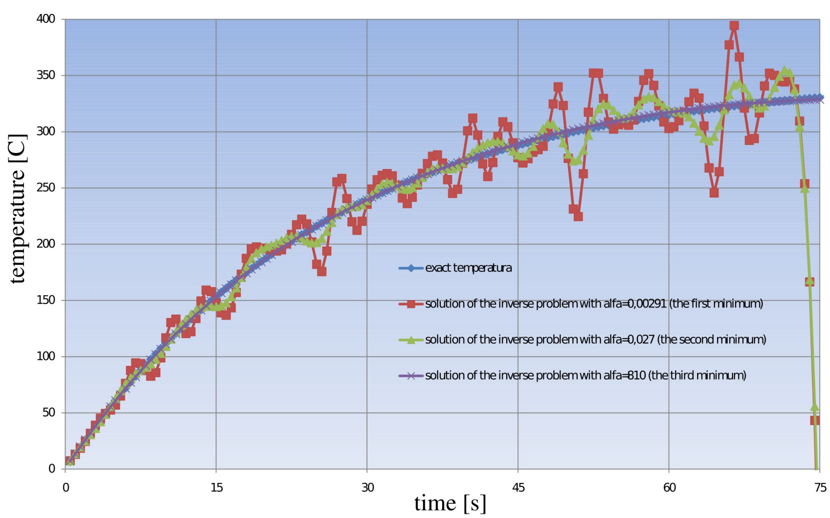

Fig. 9. Comparison of the exact temperature at $x = 0$ with the temperature obtained from the inverse problem for several appropriate values (local minimizers, compare with Fig. 6) of the regularization parameter α. The random noise level ς equals 5.0%. The case of matrix (39) is considered.

5 Concluding Remarks

The paper proposes a practical method of regularization of inverse problems where a function (a set of functions) of time has to be reconstructed. Such a function can be approximated by a sequence of segments or overlapping parabolas, and the condition of close values of the derivatives of neighboring segments/parabolas at common points can be imposed. This introduces a rigidity of the graph of the searched function. Corresponding matrix weighted penalty terms multiplied by a regularization parameter provide the stabilization of solutions. Numerical experiments show that appropriate values of the regularization parameters correspond to local minimizers of $\|\{\chi\}^{+} - \{\chi\}^{c}_{\alpha}\|_2$.

Simulations show a good agreement of reconstructed functions with exact solutions even for the high level (up to 10%) of random disturbances in measurements.

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