Body of optimal parameters in the weighted finite element method

Viktor A. Rukavishnikov[a]

[a] *Computing Center of the Far Eastern Branch of the Russian Academy of Sciences, Kim Yu Chen Str., 65, Khabarovsk, 680000, Russia*

Abstract

In [1] a weighted finite element method (WFEM) is constructed to find an approximate solution to the crack problem. We have shown that the reentrant corner 2π at the boundary of the domain does not affect the accuracy of finding the solution by this method. The approximate solution by the WFEM converges to the exact one with the rate of $O(h)$. Three control parameters affect the accuracy of finding the approximate solution by the WFEM. In this paper we define the body of optimal parameters (BOP) in the WFEM for the crack problem. The error of the found approximate solution deviates from the smallest error by no more than a predetermined value when we choose parameters from the BOP.

Keywords

corner singularity, weighted finite element method, body of optimal parameters

1. Introduction

Numerical methods for finding solutions to problems in the theory of elasticity with a singularity (tearing, sliding modes) play an essential role in fracture mechanics (see, for example, [2, 3]). For the system of Lamé equations on a nonconvex bounded polygonal domain with Dirichlet conditions, it is known [4, 5, 6, 7] that the solution to this problem can be written

$$\mathbf{u}(x) = \sum_{j=0}^{m} r_j^{\frac{\pi}{\omega_j}} \boldsymbol{\chi}(r_j, \theta_j)\Psi_j(r_j) + \boldsymbol{\psi}(x), \quad \boldsymbol{\psi} \in W_2^2(\Omega).$$

Here $\mathbf{u}(x) = (u_1, u_2)$, $\boldsymbol{\chi}(r_j, \theta_j) = (\chi_1, \chi_2)$, $\boldsymbol{\psi}(x) = (\psi_1, \psi_2)$, χ_1, χ_2 are sufficiently smooth functions, ω_j are the internal angle $\pi \le \omega_j \le 2\pi$ at the singularities p_j, (r_j, θ_j) are the polar coordinates at the point p_j and $\Psi_j(r_j)$ is the cutoff function.

A weighted finite element method (FEM) for finding an approximate solution to the problem of a crack or a Lamé system in a domain with a boundary containing an angle of 2π was proposed in [1]. This method is based on the introduction of an R_ν-generalized solution (see, for example, [8, 9, 10, 11]). The reentrant corner 2π at the boundary of the domain does not affect accuracy of finding of the approximate solution by the WFEM in compare to the classical FEM and the method with a refined mesh. The rate of convergence of an approximate solution by the WFEM to the exact one is $O(h)$ in the norm of the space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ and in the weighted energy norm [1]. The determining factor for the high accuracy of the WFEM is the correct choice of parameters: ν is exponent of the weight function in an R_ν-generalized solution, ν^* is the exponent of the weight function in the basis of the finite element method (see, for example, [12, 13, 14, 15, 16], and δ is the radius of the neighborhood in

Far Eastern Workshop on Computational Technologies and Intelligent Systems, March 2–3, 2021, Khabarovsk, Russia
vark0102@mail.ru (V.A. Rukavishnikov)
0000-0002-3702-1126 (V.A. Rukavishnikov)

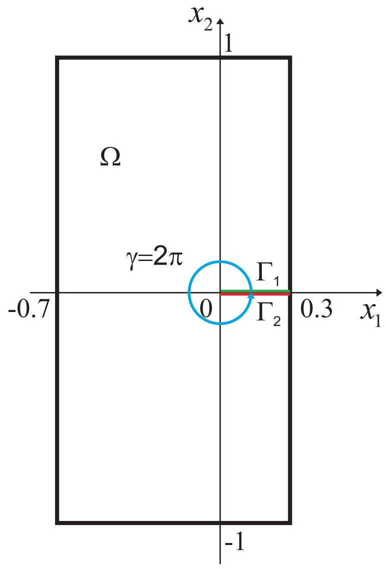


Figure 1: Rectangle domain Ω with a crack.

which the weight function is specified as the distance to the point of singularity during calculations. In this article the body of optimal parameters (BOP) for the weighted finite element method for the crack problem is determined. We have determined the body of parameters at which the error of the found approximate solution by the WFEM in the norm of the weighted Sobolev space differs from the smallest error by no more than 5%, 10%, and 15%. We noticed that the BOP depends on the dimension of mesh or mesh step.

2. R_ν-generalized solution

In [1] for finding of a displacement field $\mathbf{u} = (u_1, u_2)$ in the crack problem we considered the first boundary value problem for the Lamé system with coefficients λ and μ:

$$-\left(2\,\mathbf{div}(\mu\varepsilon(\mathbf{u})) + \mathbf{grad}(\lambda \operatorname{div} \mathbf{u})\right) = \mathbf{f}, \qquad x \in \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{q}, \qquad x \in \Gamma. \tag{2}$$

Here without loss of generality, we will assume that Ω is the rectangle shown in Fig. 1.

Let Γ be a boundary of domain Ω and $\Gamma_C \subset \Gamma$ be a crack with sides Γ_C^+ and Γ_C^+. We denote $\overline{\Omega}$ the closure of Ω, i.e. $\overline{\Omega} = \Omega \cup \Gamma$.

Comment 1. The solution of the problem (1), (2) has the form ([4])

$$\mathbf{u}(x) = r_0^{\frac{1}{2}} \boldsymbol{\chi}(r_0, \boldsymbol{\theta}_0)\boldsymbol{\Psi}_0(r_0) + \boldsymbol{\psi}(x), \tag{3}$$

where r_0 is a distance to $O(0,0)$. Therefore $\mathbf{u} \in \mathbf{W}^{1+\frac{1}{2}-\varepsilon}(\Omega)$ $(\varepsilon > 0)$ and for regular finite elements methods one obtains an order of at most $O(h^{\frac{1}{2}-\varepsilon})$, where h is the mesh step.

In [1] we proposed the weighted finite element method that allows to find an approximate solution to problem (1), (2) at a rate of $O(h)$.

Let O^δ be a disk of radius $\delta > 0$ with its centre in the point $(0,0)$, i.e. $O^\delta = \{x : (x_1^2 + x_2^2)^{\frac{1}{2}} \leq \delta \ll 1\}$ and $\Omega' = \Omega \cap O^\delta$.

Let $\rho(x)$ be a weight function that is positive everywhere, except in $O(0,0)$, and satisfies the following conditions:

a) $\rho(x) = (x_1^2 + x_2^2)^{\frac{1}{2}}$ for $x \in \overline{\Omega'}$,

b) $\rho(x) = \delta$ for $x \in \overline{\Omega} \setminus \overline{\Omega'}$.

We introduce the weighted spaces with norms:

$$\|u\|^2_{W^k_{2,\alpha}(\Omega)} = \sum_{|\lambda|\le k} \int_\Omega \rho^{2\alpha} |D^\lambda u|^2 dx, \quad \|u\|^2_{L_{2,\alpha}(\partial\Omega)} \int_{\partial\Omega} \rho^{2\alpha} u^2 ds, \quad \|u\|_{W^k_{2,0}(\Omega)} = \|u\|_{W^k_2(\Omega)}, \tag{4}$$

where $D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}}$, $\lambda = (\lambda_1, \lambda_2)$ and $|\lambda| = \lambda_1 + \lambda_2$, k is a nonnegative integer, and α is a real number.

The space $\overset{\circ}{W}{}^k_{2,\alpha}(\Omega) \subset W^k_{2,\alpha}(\Omega)$ is defined as a closure in the norm (4) of the set of infinitely differentiable and finite in Ω functions.

We say that $\varphi \in W^{1/2}_{2,\alpha}(\Gamma)$ if there exists a function $\Phi(x)$ from $W^1_{2,\alpha}(\Omega)$ such that $\Phi(x)|_\Gamma = \varphi(x)$ and

$$\|\varphi\|_{W^{1/2}_{2,\alpha}(\partial\Omega,\delta)} = \inf_{\Phi|_\Gamma = \varphi} \|\Phi\|_{W^1_{2,\alpha}(\Omega)}.$$

Let $W^1_{2,\alpha}(\Omega,\delta)$ be the set of functions satisfying the following conditions:

(a) $|D^\lambda u(x)| \le C_1 \left(\frac{\delta}{\rho(x)}\right)^{\alpha+|\lambda|}$, $x \in \Omega'$, $|\lambda| = 0, 1$, $C_1 > 0$ is a constant;

(b) $\|u\|_{L_{2,\alpha}(\Omega\setminus\Omega')} \ge C_2$, $C_2 = \text{const}$,

with norm (4).

By analogy, one can introduce sets for other spaces.

The spaces and sets for vector-functions are designated with bold letters, for example $\mathbf{W}^1_{2,\alpha}(\Omega)$.

Definition 1. [10] *Let the right-hand sides of (1), (2) satisfy the conditions*

$$\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega), \quad \mathbf{q} \in \mathbf{W}^{1/2}_{2,\beta}(\partial\Omega), \quad \beta \ge 0.$$

*A function $\mathbf{u}_\nu = (u_{\nu 1}, u_{\nu 2})$ from the space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ is called **an R_ν-generalized solution** to the problem (1), (2) if it satisfies boundary condition (2) almost everywhere on Γ and for every $\mathbf{v}$ from $\overset{\circ}{\mathbf{W}}{}^1_{\nu+\beta/2}(\Omega)$ the integral identities*

$$a_1(\mathbf{u}_\nu, v_1) = \int_\Omega \left[(\lambda+2\mu)\frac{\partial u_{\nu 1}}{\partial x_1}\frac{\partial(\rho^{2\nu} v_1)}{\partial x_1} + \mu \frac{\partial u_{\nu 1}}{\partial x_2}\frac{\partial(\rho^{2\nu} v_1)}{\partial x_2} + \lambda \frac{\partial u_{\nu 2}}{\partial x_2}\frac{\partial(\rho^{2\nu} v_1)}{\partial x_1} + \mu \frac{\partial u_{\nu 2}}{\partial x_1}\frac{\partial(\rho^{2\nu} v_1)}{\partial x_2} \right] dx =$$
$$= \int_\Omega \rho^{2\nu} f_1 v_1 dx = l_1(v_1);$$

$$a_2(\mathbf{u}_\nu, v_2) = \int_\Omega \left[\lambda \frac{\partial u_{\nu 1}}{\partial x_1}\frac{\partial(\rho^{2\nu} v_2)}{\partial x_2} + \mu \frac{\partial u_{\nu 1}}{\partial x_2}\frac{\partial(\rho^{2\nu} v_2)}{\partial x_1} + (\lambda+2\mu) \frac{\partial u_{\nu 2}}{\partial x_2}\frac{\partial(\rho^{2\nu} v_2)}{\partial x_2} + \mu \frac{\partial u_{\nu 2}}{\partial x_1}\frac{\partial(\rho^{2\nu} v_1)}{\partial x_1} \right] dx =$$
$$= \int_\Omega \rho^{2\nu} f_2 v_2 dx = l_2(v_2);$$

holds for any fixed value of ν satisfying the inequality $\nu \geq \beta$.

Comment 2. We notice that an R_ν-generalized solution has a sheaf of solutions in the neighborhood of the singularity point if it is defined in the weighted space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ and does not take into account the additional properties of this solution (see, for example, [17]). In [10] we proved the uniqueness of an R_ν-generalized solution if it is defined in the set $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega,\delta)$.

An R_ν-generalized solution satisfies conditions (a), (b) of the set $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega,\delta)$. This follows from the asymptotic of the solution to problem (1), (2) (see (3)). We use the "special" properties of functions from this set additionally. At the same time, we do not refuse to use the properties of the space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ (the presence of a zero element, etc.).

Comment 3. We proved that an R_ν-generalized solution is the same for different ν (see [10]).

Comment 4. In contrast to the weak solution of problem (1), (2), the weight function is introduced into the definition of an R_ν-generalized solution. This allows us to suppress the influence of the singularity on the regularity of the solution. In [18] we proved that an R_ν-generalized solution of a boundary value problem for a second-order elliptic equation belongs to the weighted space $W^2_{2,\nu+\beta/2}(\Omega)$. Subsequently this made it possible to establish the convergence of the approximate solution to the R_ν-generalized solution with a rate $O(h)$ ([19]).

3. Weighted finite element method

The weighted finite element method for finding an approximate an R_ν-generalized solution of problem (1), (2) was constructed in [1]. Here we briefly describe construction of the WFEM.

We perform a quasi-uniform triangulation of the domain $\overline{\Omega}$ (see Fig. 2). Let K is the union of all the triangles K_i, $i = 1,\dots,n$; h is the maximal length of the sides of the triangles and it called mesh step. The vertices P_i, $i = 1,\dots,M$ of the triangles K are nodes of the triangulation, $\{P^M\} = \{P_1,\dots,P_M\}$ and the point $O \in P^M$. Let $P = \{P_k\}_{k=1}^{k=N}$ is the set of internal triangulation nodes.

To each node $P_i \in P$ we assign the weighted function

$$\hat{\psi}_i = \rho^{\nu^*}(x)\varphi_i, \quad i = 1,\dots,N,$$

where φ_i is linear function on each triangle K, equal to 1 at the node P_i and zero at all the other nodes, ν^* is a real number.

We introduce weighted vector basis $\{\psi_k(x)\}_{k=1}^{k=2N}$, where

$$\boldsymbol{\psi}_k(x)\begin{cases} (\hat{\psi}_i(x), 0), k = 2i-1, \\ (0, \hat{\psi}_i(x)), k = 2i, \end{cases} \qquad i = 1,...,N.$$

We denote by $\mathbf{V}^h$ the linear span $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{k=2N}$. In $\mathbf{V}^h$ we denote the subset $\mathring{\mathbf{V}}^h = \{\mathbf{v} \in \mathbf{V}^h : \mathbf{v}(P_i) = 0, P_i \in \Gamma\}$.

Definition 2. *A function $\mathbf{u}^h_\nu$ in the space $\mathbf{V}^h$ is called an **approximate R_ν-generalized solution** of the problem (1), (2) by the weighted finite element method if it satisfies the boundary condition (2) for mesh nodes $P_i \in \Gamma$ and the integral identity*

$$a(\mathbf{u}^h_\nu, \mathbf{v}^h) = l(\mathbf{v}^h)$$

holds for all $\mathbf{v}^h \in \mathring{\mathbf{V}}^h$ and $\nu \geq \beta$. Here $a(\mathbf{u}^h_\nu, \mathbf{v}^h) = (a_1(\mathbf{u}^h_\nu, v^h_1), a_2(\mathbf{u}^h_\nu, v^h_2))$, $l(\mathbf{v}) = (l_1(v^h_1), l_2(v^h_2))$.

An approximate solution will be found in the form

$$\mathbf{u}^h_\nu = \sum_{k=1}^{2N} d_k \boldsymbol{\psi}_k(x),$$

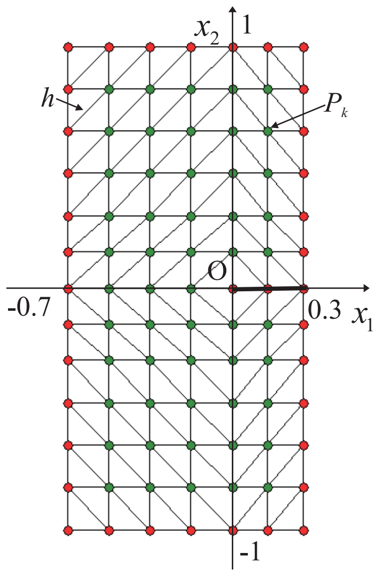


Figure 2: Triangulation of domain Ω.

here $d_k = \rho^{-\nu^*}(P_{[(k+1)/2]})c_k$, $c_k = \begin{cases} u^h_{\nu,1}(P_{[(k+1)/2]}), k = 2i-1 \\ u^h_{\nu,2}(P_{[(k+1)/2]}), k = 2i \end{cases}$, $i = 1, \ldots, N$, $[(k+1)/2]$ is an integer part of number $(k+1)/2$.

Comment 5. Note that we have introduced into the basis the weight function raised to some power. The weight basis and an R_ν-generalized solution made it possible to find an approximate solution without loss of accuracy on quasi-uniform grids.

We proved that the approximate R_ν-generalized solution by the weighted FEM converges to the exact one with the first rate with respect to the mesh step h [19, Theorem 8].

In [1] a numerical analysis was carried out for one model problem on grids of large and small dimensions.

We have obtained experimentally confirmation of the convergence rate of the approximate solution to the exact one $O(h)$ in the norm of the space $\mathbf{W}^1_{2,\nu}(\Omega)$ and in the energy norm. In addition, the smallness of the absolute error (10^{-7}) in the overwhelming number of grid nodes was established.

4. Body of optimal parameters

4.1. Algorithm for determining BOP on grids of various dimensions

For calculation of the approximate R_ν-generalized solution by the weighted finite element method we need to set the parameters ν, ν^*, δ. These parameters can be arbitrary if they satisfy conditions of the theorem on the existence and uniqueness of the R_ν-generalized solution and correspond to the asymptotic properties of the solution. But if you want to find an approximate solution to the problem with the smallest error, then these parameters should be close to optimal. Currently, there is no algorithm for theoretical determination of such parameters. For problem (1), (2) we will find them experimentally.

Consider two model problems in the domain Ω:

(A) Boundary value problem (1), (2) with a solution containing only a singular component

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left(1 - \frac{\lambda}{\lambda+\mu} + \sin^2\left(\frac{\theta}{2}\right)\right),$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left(2 - \frac{\lambda}{\lambda+\mu} + \cos^2\left(\frac{\theta}{2}\right)\right),$$

Lamé coefficients are $\lambda = 576.923, \mu = 384.615$, and stress intensity factor $K_I = 1.611$.

(B) Boundary value problem (1), (2) with a solution containing a singular and a regular component from the space $W_2^2(\Omega)$

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left(1 - \frac{\lambda}{\lambda+\mu} + \sin^2\left(\frac{\theta}{2}\right)\right) + r^2,$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left(2 - \frac{\lambda}{\lambda+\mu} + \cos^2\left(\frac{\theta}{2}\right)\right) + r^2.$$

Let us find for problems (A) and (B) the parameters ν, ν^*, δ, which allow us to calculate an approximate solution by the weighted finite element method with the best accuracy on quasiuniform meshes of various dimensions. In Ω we built meshes with a step $h = 0.062, 0.031, 0.015, 0.0077, 0.0038, 0.0019$ and determined the BOP for each of these meshes.

The set of optimal parameters will be discrete, as we form it from the results of numerical experiments carried out for given fixed values ν, ν^*, δ.

We chose ν^* equal to $0, 0.1, 0.2, 0.3, 0.4, 0.49$. The values of ν were selected from the interval $[0.5, 5.5]$ with a step of 0.1. The radius of the δ-neighborhood Ω' was equated to $h, 2h, 3h, \ldots$. Calculations were stopped or later disregarded when the error between the exact solution and the found approximate solution became larger than specified limiting error. The relative error was determined for all grids and parameters of WFEM in the weighted Sobolev norm and weighted energy norm with fixed and predetermined parameters $\bar{\nu} = 2.2$, $\bar{\delta} = 0.062$. Note that when choosing other parameters $\bar{\nu}$ and $\bar{\delta}$, there were no significant changes in the results.

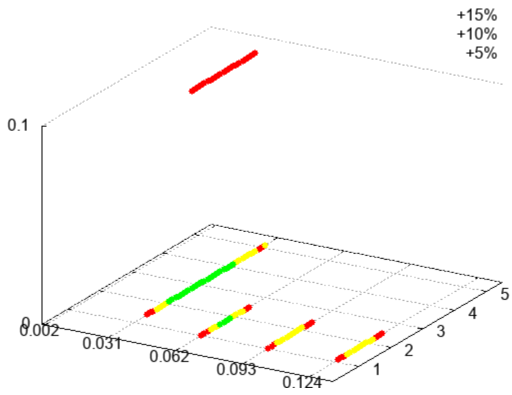
For each problem (A), (B) and each mesh we determined three parameters ν, ν^*, δ for which the relative error in the weighted Sobolev norm and weighted energy norm was the smallest. In addition, we formed sets of parameters $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$ and $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$ at which the relative errors differed from the best error by no more than 5%(6.5%), 10% and 15%.

Comment 6. The ratio of the smallest errors was exactly two in both weighted norms on adjacent meshes for problems (A) and (B). This corresponds to a theoretical estimate of the convergence rate.

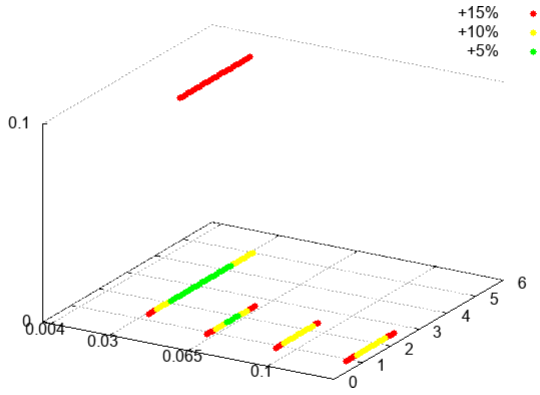
For each mesh the body of optimal parameters (BOP) is $T_i = T_i^{(A)} \cap T_i^{(B)}$, $i = 1, 2, 3$. In addition, we have determined triples of parameters ν, ν^*, δ, which allow us to find an approximate solution with an error that differs from the best error by no more than 6.5% on all meshes simultaneously.

4.2. Results of numerical experiments

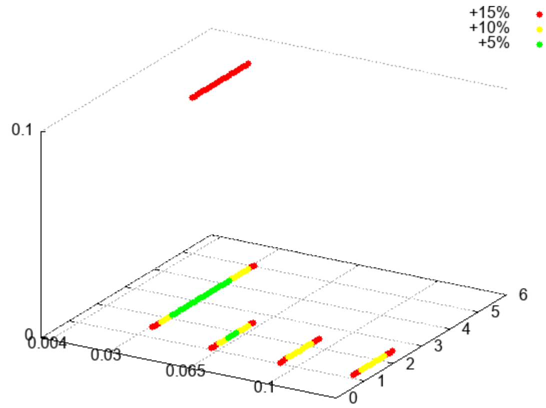
Figures 3, 4, 5, 6 show the parameters ν, ν^*, δ at which the errors differ from the best error by no more than 5% (green), 10% (yellow), 15% (red) at $h = 0.031, 0.015$ and $h = 0.0038, 0.0019$ respectively. We present the results for tasks A and B in Figures 3a – 6a and 3b – 6b, respectively. Figures 3c – 6c depict the sets T_i, $i = 1, 2, 3$. In Tables 1 and 2 we indicated the intervals of the parameters ν, ν^*, δ of the BOP at which the relative error in the norm of the weight space deviates from the best error by no more than the indicated values for $h = 0.031, 0.015$ and $h = 0.0038, 0.0019$ respectively.



(3a)

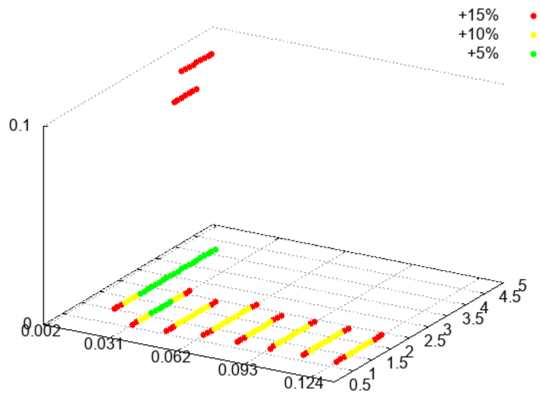


(3b)

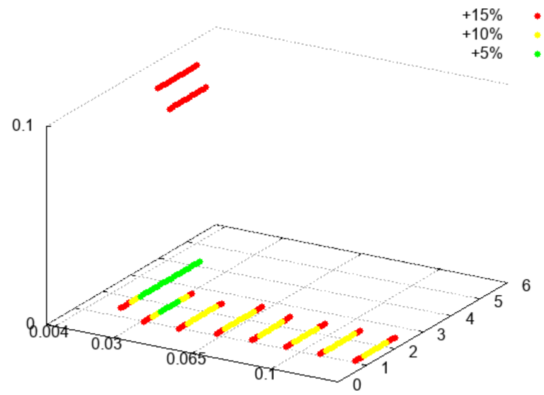


(3c)

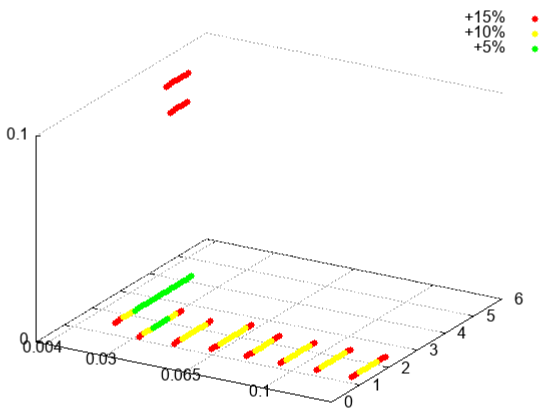
Figure 3: (3a) the sets $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$; (3b) the sets $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$; (3c) the sets $T_i = T_i^A \cap T_i^B, i = 1, 2, 3$ for the mesh with step $h = 0.031$.



(4a)



(4b)

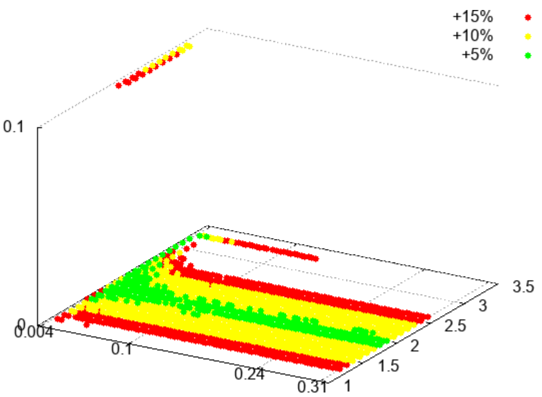


(4c)

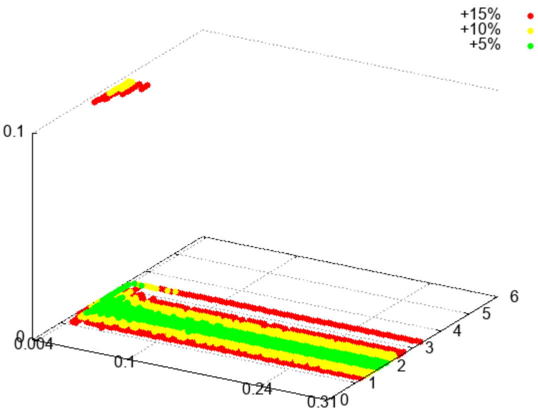
Figure 4: (4a) the sets $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$; (4b) the sets $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$; (4c) the sets $T_i = T_i^A \cap T_i^B, i = 1, 2, 3$ for the mesh with step $h = 0.015$.

Table 1
Optimal parameters with a given error for the meshes with steps $h = 0.031, 0.015$.

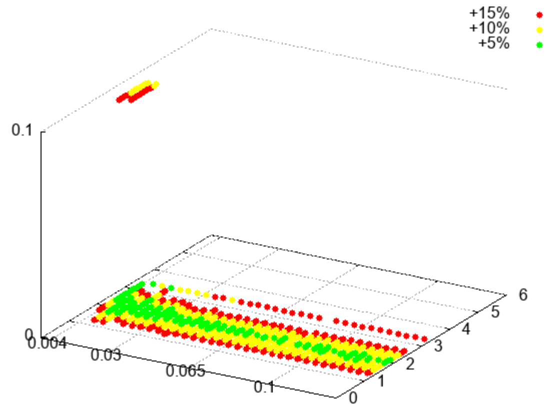
percent of error	δ	ν	ν^*	percent of error	δ	ν	ν^*
+5%	0.03094	2.1..4.0	0.0	+5%	0.01547	2.2..4.0	0.0
	0.06187	1.6..1.8	0.0		0.03094	1.6..2.1	0.0
+10%	0.03094	1.7..4.8	0.0	+10%	0.01547	1.8..4.2	0.0
	0.06187	1.3..2.2	0.0		0.03094	1.3..2.4	0.0
	0.09281	1.3..2.1	0.0		0.0464-0.12374	1.4..2.2	0.0
	0.12374	1.4..2.2	0.0	+15%	0.01547	1.5..4.2	0.0
+15%	0.03094	1.4..5.0	0.0		0.01547	3.3..4.1	0.1
	0.03094	2.8..4.8	0.1		0.03094	1.1..2.6	0.0
	0.06187	1.0..2.5	0.0		0.03094	2.2..2.8	0.1
	0.09281	1.0..2.4	0.0		0.0464-0.12374	1.2..2.4	0.0
	0.12374	1.1..2.5	0.0				



(5a)

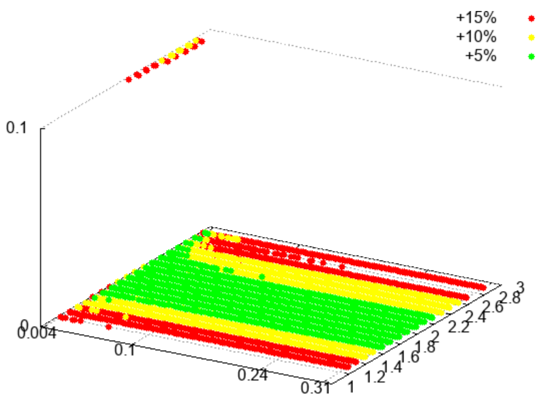


(5b)

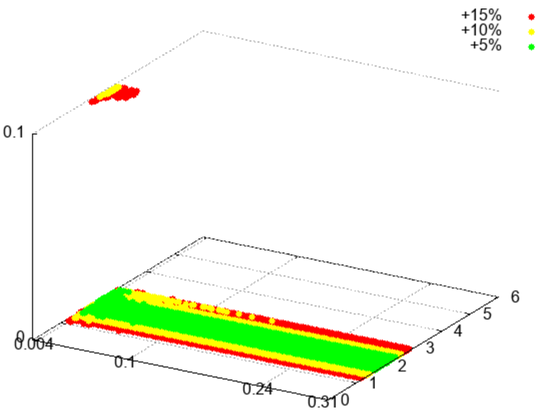


(5c)

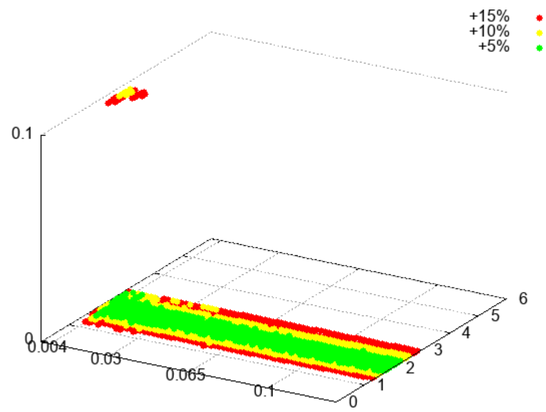
Figure 5: (5a) the sets $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$; (5b) the sets $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$; (5c) the sets $T_i = T_i^A \cap T_i^B$, $i = 1, 2, 3$ for the mesh with step $h = 0.0038$.



(6a)



(6b)



(6c)

Figure 6: (6a) the sets $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$; (6b) the sets $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$; (6c) the sets $T_i = T_i^A \cap T_i^B$, $i = 1, 2, 3$ for the mesh with step $h = 0.0019$.

Table 2
Optimal parameters with a given error for the meshes with steps $h = 0.0038, 0.0019$.

percent of error	δ	ν	ν^*	percent of error	δ	ν	ν^*
+5%	0.00387	2.3..3.2	0.0	+5%	0.00193	2.3..2.8	0.0
	0.00773	1.7..2.5	0.0		0.00387	1.6..2.6	0.0
	0.0116	1.7..2.3	0.0		0.00387	2.9..2.9	0.0
	0.01547	1.9..2.6	0.0		0.0058	1.7..2.5	0.0
	0.01934-0.02707	1.8..2.2	0.0		0.00773	1.8..2.9	0.0
	0.03094-0.3	1.9..2.1	0.0		0.00967-0.3	1.8..2.1	0.0
+10%	0.00387	2.0..3.2	0.0	+10%	0.00193	1.9..2.8	0.0
	0.00773	1.4..2.7	0.0		0.00387	1.4..2.9	0.0
	0.00773	3.3..3.3	0.0		0.00387	2.4..2.8	0.1
	0.00773	2.5..3.1	0.1		0.0058	1.4..2.7	0.0
	0.0116	1.4..2.6	0.0		0.0058	2.9..2.9	0.0
	0.0116	3.3..3.3	0.0		0.0058	2.5..2.7	0.1
	0.0116	3.0..3.1	0.1		0.00773	1.6..2.9	0.0
	0.01547	1.6..2.9	0.0		0.00967	1.5..2.7	0.0
	0.01547	3.3..3.3	0.0		0.00967	2.9..2.9	0.0
	0.01934-0.3	1.5..2.4	0.0		0.0116- 0.3	1.5..2.5	0.0

We present the values of the parameters at which the deviation of the relative error from the best error does not exceed 5%, 5.5% and 6% for problem A on the mesh with a step $h = 0.0038$ in Fig. 7.

5. Discussion of the results. Conclusion

In this paper we defined the body of optimal parameters in the weighted finite element method to find an approximate solution to the crack problem with high accuracy. Finding the BOP is based on a series of numerical experiments. We used the knowledge about the asymptotic behavior of the solution in the neighborhood of the singularity point and the conditions on the input data ν, δ of the existence and uniqueness theorem for the R_ν-generalized solution. The results of the experiments led to the following conclusions:

1. The proposed approach allows us to determine the BOP for the weighted finite element method (Figure 3–6, Table 1,2).
2. BOP depends on the dimension of the mesh (mesh step).
3. With a small deviation in the choice of parameters from the best parameters in WFEM, the relative error in the norm of the Sobolev weight space grows slightly (see Fig. 7). This indicates the stability of the process, i.e., a small change in the control parameters corresponds to a small increase in the relative error of the approximate solution.
4. If we choose the parameters $\delta = 0.062$, $\nu = 2.0$, $\nu^* = 0$ then the error does not exceed 6.75% of the best value the error on all meshes simultaneously. In our opinion this fact is not important. The optimal parameters for carrying out calculations should be chosen depending on the dimension of the mesh (mesh step).
5. The proposed method with the found BOP can be used for calculating engineering problems.

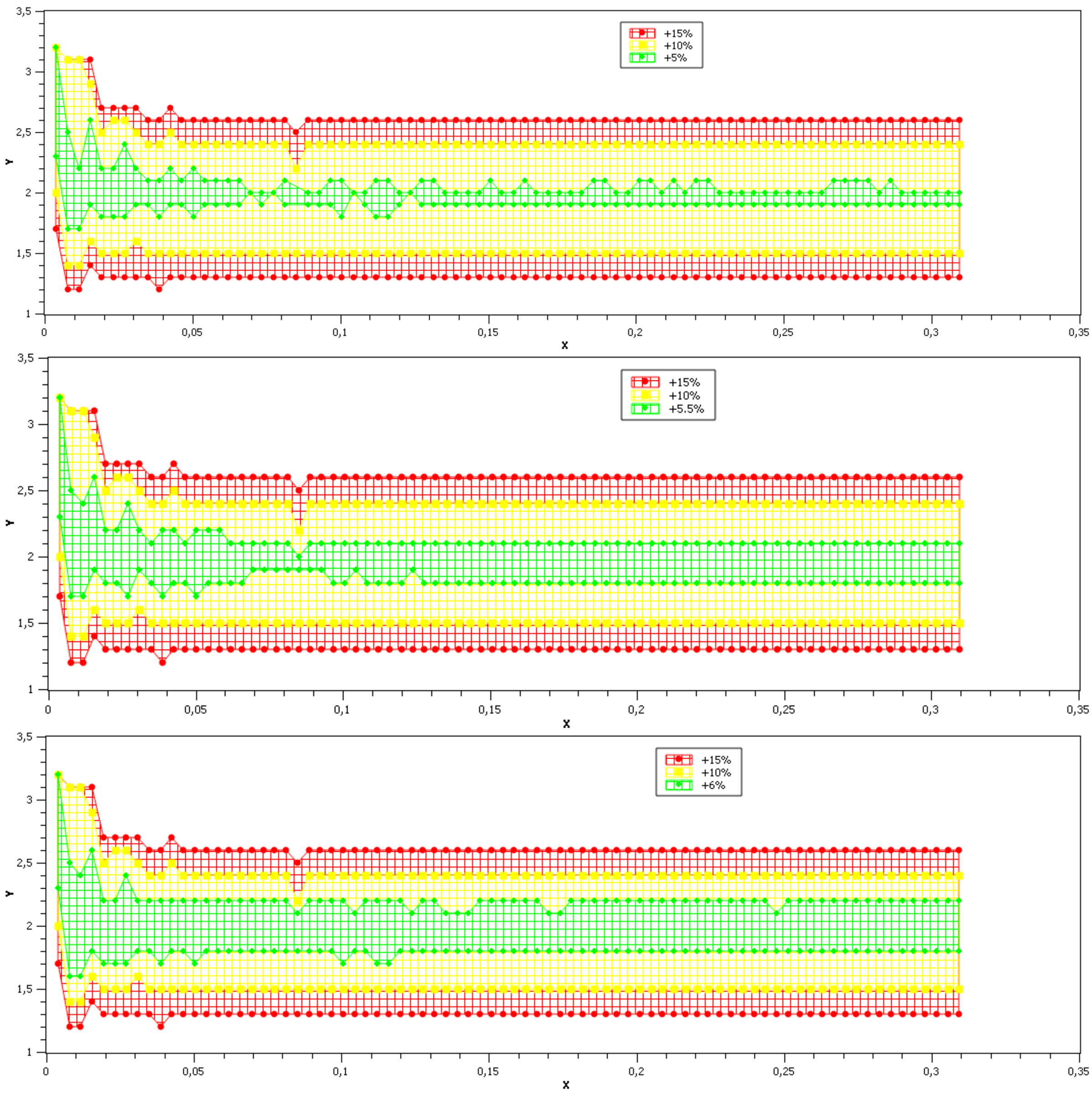


Figure 7: Parameter values at which the deviation of the relative error from the best error does not exceed 5%, 5.5%, and 6% for problem A on the mesh with a step $h = 0.0038$.

Acknowledgments

I would like to thank Andrei Mosolapov and Artem Zubkov for their help in organizing and conducting numerical experiments.

The reported study was supported by RSF according to the research project No. 21-11-00039. Computational resources for the numerical experiments were provided by the Shared Services Center "Data Center of FEB RAS".

References

[1] V. A. Rukavishnikov, A. O. Mosolapov, E. I. Rukavishnikova, Weighted finite element method for elasticity problem with a crack, Computers and Structures 243 (2021) 106400. doi:10.1016/j.compstruc.2020.106400.

[2] E. E. Gdoutos, Fracture Mechanics Criteria and Applications, volume 10 of *Engineering Applications of Fracture Mechanics*, Kluwer Academic Publishers, Dordrecht, 1990.

[3] B. Szabó, I. Babuška, Finite element analysis, John Wiley & Sons, New York, NY, 1991.

[4] V. A. Kondrat'ev, Boundary-value problems for elliptic equations in domains with conical or angular points, Trans. Moscow Math. Soc. 16 (1967) 227–313.

[5] V. G. Mazya, B. A. Plamenevskij, L_p-estimates of solutions of elliptic boundary value problems in domains with edges, Trans. Moscow Math. Soc. 1 (1980) 49–97.

[6] P. Grisvard, Boundary Value Problems in Non-Smooth Domains, Pitman, London, 1985.

[7] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, volume 1341 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1988. doi:10.1007/BFb0086682.

[8] V. A. Rukavishnikov, E. V. Kuznetsova, The R_ν-generalized solution of a boundary value problem with a singularity belongs to the space $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega, \delta)$, Differential Equations 45 (2009) 913–917. doi:10.1134/S0012266109060147.

[9] V. A. Rukavishnikov, On the existence and uniqueness of an R_ν-generalized solution of a boundary value problem with uncoordinated degeneration of the input data, Doklady Mathematics 90 (2014) 562–564. doi:10.1134/S1064562414060155.

[10] V. A. Rukavishnikov, E. I. Rukavishnikova, Existence and uniqueness of an R_ν-generalized solution of the Dirichlet problem for the Lamé system with a corner singularity, Differential Equations 55 (2019) 832–840. doi:10.1134/S0012266119060107.

[11] V. A. Rukavishnikov, E. I. Rukavishnikova, On the Dirichlet problem with corner singularity, Mathematics 8 (2020) 1870. doi:10.3390/math8111870.

[12] V. A. Rukavishnikov, A. Y. Bespalov, An exponential rate of convergence of the finite element method for the Dirichlet problem with a singularity of the solution, Doklady Mathematics 62 (2000) 266–270.

[13] V. A. Rukavishnikov, E. V. Kuznetsova, A finite element method scheme for boundary value problems with noncoordinated degeneration of input data, Numerical Analysis and Applications 2 (2009) 250–259. doi:10.1134/S1995423909030069.

[14] V. A. Rukavishnikov, E. I. Rukavishnikova, Numerical method for Dirichlet problem with degeneration of the solution on the entire boundary, Symmetry 11 (2019) 1455. doi:10.3390/sym11121455.

[15] V. A. Rukavishnikov, A. V. Rukavishnikov, New numerical method for the rotation form of the Oseen problem with corner singularity, Symmetry 11 (2019) 54. doi:10.3390/sym11010054.

[16] V. A. Rukavishnikov, O. P. Tkachenko, Dynamics of a fluid-filled curvilinear pipeline, Applied Mathematics and Mechanics 39 (2018) 905–922. doi:10.1007/s10483-018-2338-9.

[17] V. A. Rukavishnikov, The Dirichlet problem for a second-order elliptic equation with noncoordinated degeneration of the input data, Differential Equations 32 (1996) 406–412.

[18] V. A. Rukavishnikov, E. V. Kuznetsova, Coercive estimate for a boundary value problem with noncoordinated degeneration of the data, Differential Equations 43 (2007) 550–560. doi:10.1134/S0012266107040131.

[19] V. A. Rukavishnikov, Weighted FEM for two-dimensional elasticity problem with corner singularity, Lecture Notes in Computational Science and Engineering 112 (2016) 411–419. doi:10.1007/978-3-319-39929-4_39.