Mathematical and computer modeling of endovenous laser treatment

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Abstract
An initial-boundary value problem for quasilinear equations of radiative-conductive heat transfer, simulating the process of endovenous laser ablation, is considered. The unique solvability of the problem is proved and an algorithm for finding its solution is proposed. The efficiency of the algorithm is illustrated by numerical experiments.

Keywords
numerical simulation, endovenous laser ablation, quasilinear equations of heat transfer

1. Introduction

The endovenous laser ablation (EVLA) is an effective and popular treatment of varicose veins. During EVLA a laser optical fiber is introduced into a damaged vein. Afterwards, the optical fiber transmitting laser radiation is drawn out of the vein. The fiber tip emitting radiation is usually coated with a carbonized layer, which splits the laser energy, such that a part of the energy is absorbed by the carbonized layer with the release of heat, the other part is spent on radiation. The heat from the carbonized layer is transmitted into blood by the conductive heat transfer. Heat transfer is significantly intensified by the flow of bubbles formed on the hot fiber tip. The radiation entering into the blood and the surrounding tissue is partially absorbed with the release of heat. As a result, the heat energy generated by different mechanisms causes essential heating the vein, that leads to its obliteration (closure).

Optimization of radiation parameters during EVLA ensures successful vein obliteration with minimum frequency and severity of complications. In other words, optimal radiation must provide a rather high temperature inside the vein for its obliteration and simultaneously the generated temperature field must be relatively safe for the tissue surrounding the vein. The principal parameters influencing the efficiency and safety of the laser ablation procedure are the laser power, radiation wavelength, pullback speed of the optical fiber, and the ratio between the laser powers spent on radiation and fiber tip (carbonized layer) heating. As a rule, the laser ablation is performed by a radiation with a

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wavelength from 810 to 1950 nm. The sufficiently widely used ranges of the pullback speed of the fiber and laser radiation power are 1–3 mm/s and 5–15 W, respectively.

The classic model describing the interaction between optical radiation and a biological tissue is a reaction-diffusion model including the radiation transfer and heat conduction equations [1, 2]. Correspondingly, the calculation of radiation and temperature fields involves finding the bulk density of absorbed radiation energy from the radiation transfer equation and subsequent solving the heat conductivity equation, in which the absorbed radiation energy form heat sources. The theoretical and numerical analysis of reaction-diffusion models describing various phenomena in physics, biology, and ecology can be found in [3, 4, 5, 6]. The reaction-diffusion models of heat transfer accounting the radiation effects are studied in [7, 8]. In [9, 10, 11, 12], to simulate EVLA, the authors apply the reaction-diffusion heat transfer equation, where the bulk density of the absorbed energy of radiation is calculated using an explicit formula for intensity of radiation induced by a point source in infinite homogeneous medium.

The principal effects, which are usually taken into account in EVLA modeling, are the conductive heat transfer, radiation transfer and absorption with the release of heat, and heat transfer by the flow of bubbles formed at the hot fiber tip. In the works [10, 11, 12], on the base of an estimation of experimental data, heat transfer by the flow of bubbles is modelled by the use of a piecewise constant heat conductivity coefficient depending on the temperature as follows: when the temperature at a certain point reaches 95°C and higher, the heat conductivity coefficient increases by 200 times. We will use this approach in the model of the reaction-diffusion describing process of EVLA. The corresponding quasilinear initial-boundary value problem of EVLA is studied in the current work. The unique solvability of the problem is proved and an algorithm for finding its solution is proposed. The efficiency of the algorithm is illustrated by numerical examples. Theoretical and numerical analysis of the considered model are also conducted in [13, 14], where extremal problems for the quasilinear model of EVLA are studied.

2. Mathematical model of endovenous laser ablation

Let us formulate an initial-boundary value problem for the following equations modeling process of EVLA in a bounded three-dimensional domain Ω with the boundary $\Gamma = \partial\Omega$:

$$\sigma\partial\theta/\partial t - \operatorname{div}(k(\theta)\nabla\theta) - \mu_a\varphi = u_1\xi, \quad -\operatorname{div}(\alpha\nabla\varphi) + \mu_a\varphi = u_2\xi, \quad x \in \Omega, \quad 0 < t < T, \tag{1}$$

$$k(\theta)\partial_n\theta + \gamma(\theta - \theta_b) = 0, \quad \alpha\partial_n\varphi + 0.5\varphi = 0 \ \text{ on } \Gamma, \tag{2}$$

$$\theta|_{t=0} = \theta_0. \tag{3}$$

Here, θ is the temperature, φ is the radiation intensity averaged over all directions, α is the diffusion coefficient for optical radiation, μ_a is the absorption coefficient, $k(\theta)$ is the thermal conductivity coefficient, $\sigma(x, t)$ is the product of the specific heat capacity and the density of the medium, γ is the heat transfer coefficient, u_1 describes the source power used on heating the fiber tip, and u_2 is the source power going to the radiation, ξ is the characteristic function of the fiber tip area divided by the volume of the fiber tip. The functions θ_b, θ_0 define the boundary and initial temperature distributions, respectively. By ∂_n we denote the derivative in the direction of outward normal $\mathbf{n}$ to the boundary Γ.

3. Solvability of the initial-boundary value problem

3.1. Quasilinear equation of heat transfer

In what follows, we assume that Ω is a Lipschitz bounded domain, $\Gamma = \partial\Omega$, $Q = \Omega\times(0, T)$, $\Sigma = \Gamma\times(0, T)$. By L^p, $1 \le p \le \infty$, we denote Lebesgue space, through H^1 the Sobolev space W_2^1, and by $L^p(0, T; X)$ the Lebesgue space of functions of the class L^p, defined on $(0, T)$, with values in the Banach space X.

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$, by V' we denote the dual space of V. The space H is identified with the space H', so that $V \subset H = H' \subset V'$. We denote by $\|\cdot\|$ the standard norm in H, and by (w, v) the value of the functional $w \in V'$ at the element $v \in V$, which coincides with the inner product of H if $w \in H$.

Let us consider the problem:

$$\sigma\partial\theta/\partial t - \operatorname{div}(k(\theta)\nabla\theta) = f, \quad x \in \Omega, \quad 0 < t < T, \tag{4}$$

$$k(\theta)\partial_n\theta + \gamma(\theta - \theta_b)|_\Gamma = 0, \quad \theta|_{t=0} = \theta_0. \tag{5}$$

We will assume that the initial data satisfy conditions:

(i) $0 < \sigma_0 \le \sigma \le \sigma_1$, $|\partial\sigma/\partial t| \le \sigma_2$, $\sigma_j = Const.$
(ii) $0 < k_0 \le k(s) \le k_1$, $|k'(s)| \le k_2$, $s \in \mathbb{R}$, $k_j = Const.$
(iii) $f \in L^2(0, T; V')$, $\theta_0 \in H.$
(iv) $\gamma \in L^\infty(\Gamma)$, $\gamma \ge \gamma_0 = Const > 0$, $\theta_b \in L^\infty(\Sigma).$

We define a nonlinear operator $A : V \longrightarrow V'$, using the equality valid for any $\theta, v \in V$:

$$(A(\theta), v) = (k(\theta)\nabla\theta, \nabla v) + \int_\Gamma \gamma\theta v d\Gamma = (\nabla h(\theta), \nabla v) + \int_\Gamma \gamma\theta v d\Gamma.$$

Here, $h(s) = \int_0^s k(r)dr$. Let $g \in L^\infty(0, T; V')$, $(g, v) = \int_\Gamma \gamma\theta_b v d\Gamma$.

In what follows, as the inner product in V, we will use the bilinear form $(u, v)_V = (\nabla u, \nabla v) + \int_\Gamma uv d\Gamma$. The corresponding norm is equivalent to the standard norm of the space V.

The following weak formulation is valid for problem (4)–(5).

Definition. *Function $\theta \in L^2(0, T; V)$ is a weak solution of the problem (4)–(5) if $\sigma\theta' \in L^2(0, T; V')$ and*

$$\sigma\theta' + A(\theta) = \eta, \quad \theta(0) = \theta_0. \tag{6}$$

Hereinafter, $\theta' = d\theta/dt$, $\eta = f + g$.

Note that $(\sigma\theta)' = \sigma\theta' + \theta\partial\sigma/\partial t \in L^2(0, T; V')$. Then $\sigma\theta \in L^2(0, T; V')$ and hence $\sigma\theta \in C([0, T]; V')$. Therefore, the initial condition $\theta(0) = \theta_0$ makes sense.

Theorem 1. *Let conditions (i)–(iii) hold. Then the problem (4)–(5) is solvable.*

Proof. Let us define the Galerkin approximations θ_m of a solution of the problem (4)–(5) and derive a priori estimates which are necessary for proving the solvability. In the space V, we consider an orthonormal in H basis $w_1, w_2, \dots$, $V_m = \operatorname{span}\{w_1, ..., w_m\}$. Let $\theta_m(t) \in V_m$, $t \in (0, T)$, be a solution of the following Cauchy problem:

$$(\sigma\theta_m' + A(\theta_m) - \eta, \zeta) = 0 \quad \forall\zeta \in V_m, \quad \theta_m|_{t=0} = \theta_{0m}. \tag{7}$$

Here, θ_{0m} is the orthogonal projection in H of the function θ_0 onto the subspace V_m.

Let us derive a priori estimates which are necessary for proving the solvability. Put $\zeta = \theta_m$ in (7) and integrate with respect to time from 0 to t. Then

$$\frac{1}{2}\|\sqrt{\sigma}\theta_m\|^2 + \int_0^t \left(k(\theta(s))\nabla\theta_m(s), \nabla\theta_m(s)) + \int_\Gamma \gamma\theta_m^2 d\Gamma \right) ds =$$

$$\frac{1}{2}\|\sqrt{\sigma|_{t=0}}\theta_{0m}\|^2 + \frac{1}{2}\int_0^t (\sigma_t\theta_m, \theta_m)ds + \int_0^t (\eta, \theta_m)ds.$$

Let $\nu = \min\{k_0, \gamma_0\} > 0$. Since

$$(\eta, \theta_m) \le \frac{1}{2\nu}\|\eta\|_{V'}^2 + \frac{\nu}{2}\|\theta_m\|_V^2,$$

then, accounting for the conditions (i), (ii), (iv), and the definition of the norm in V, we derive the inequality

$$\sigma_0\|\theta_m(t)\|^2 + \nu\int_0^t \|\theta_m(s)\|_V^2 ds \le K + \sigma_2\int_0^t \|\theta_m(s)\|^2 ds. \tag{8}$$

Here, $K = \sigma_1\|\theta_0\|^2 + \nu^{-1}\|\eta\|_{V'}^2$. By the Gronwall's inequality, for $\|\theta_m(t)\|^2$, we obtain the estimate

$$\|\theta_m(t)\|^2 \le \frac{K}{\sigma_0}\exp\frac{\sigma_2 t}{\sigma_0} \quad \text{a.e. on } (0, T). \tag{9}$$

The resulting inequality allows one to estimate the right-hand side in (8) and therefore

$$\nu\int_0^T \|\theta_m(s)\|_V^2 ds \le K\exp\frac{\sigma_2 T}{\sigma_0}. \tag{10}$$

Thus,

$$\|\theta_m\|_{L^\infty(0,T;H)} \le C, \quad \|\theta_m\|_{L^2(0,T;V)} \le C, \quad \|h(\theta_m)\|_{L^2(0,T;V)} \le C. \tag{11}$$

Here and below, in the proof of the theorem, by C we denote constants independent of m. The obtained estimates (11) allow us to assert that passing, if necessary, to a subsequence, there is a function θ such that

$$\theta_m \longrightarrow \theta \quad \text{weakly in } L^2(0,T;V), \quad * -\text{weakly in } L^\infty(0,T;H),$$
$$h(\theta_m) \longrightarrow \chi \quad \text{weakly in } L^2(0,T;V). \tag{12}$$

Convergence results (12) are sufficient for passing to the limit as $m \longrightarrow \infty$ in the system (7) and proving that the limit function $\theta \in L^2(0,T;V)$ is such that $\sigma\theta' \in L^2(0,T;V')$, and the equality

$$(\sigma\theta', v) + (\nabla\chi, \nabla v) + \int_\Gamma \gamma\theta v d\Gamma = (\eta, v) \quad \forall v \in V$$

and the initial condition hold. To prove that θ is a weak solution of the problem (4)–(5), it is sufficient to check that the equality $\chi = h(\theta)$ holds. We obtain an estimate guaranteeing the compactness of the sequence θ_m in $L^2(Q)$. In the system (7), we put $\zeta = \theta_m(t) - \theta_m(s)$ and integrate with respect to t on the interval $(s, s+\delta)$ and with respect to s on $(0, T-\delta)$, assuming that $\delta > 0$ is small enough. Then

$$\frac{1}{2}\int\limits_0^{T-\delta} \|\sqrt{\sigma}(\theta_m(s+\delta) - \theta_m(s))\|^2 ds = \int\limits_0^{T-\delta}\int\limits_s^{s+\delta} c_m(t,s)\,dt\,ds,$$

where

$$c_m(t,s) = (\nabla h(\theta_m(t)), \nabla(\theta_m(s) - \theta_m(t))) + \int\limits_\Gamma \gamma\theta_m(t)(\theta_m(s) - \theta_m(t))d\Gamma -$$

$$(\eta(t) + \frac{1}{2}\sigma_t(\theta_m(s) - \theta_m(t)), \theta_m(s) - \theta_m(t)).$$

Accounting for the non-negativity of $(\nabla h(\theta_m(t)), \nabla\theta_m(t)) + \int\limits_\Gamma \gamma\theta_m^2(t)d\Gamma$, we obtain the inequality

$$c_m(t,s) \leq \nu_1|(\theta_m(t), \theta_m(s))_V| + (\eta(t), \theta_m(s) - \theta_m(t)) + \frac{1}{2}\sigma_2\|\theta_m(s) - \theta_m(t))\|^2.$$

Here, $\nu_1 = \max\{k_1, \|\gamma\|_{L^\infty(\Gamma)}\}$. Therefore, taking into account the continuity of the embedding $V \subset H$, we have

$$c_m(t,s) \leq C\left(\|\eta(t)\|^2_{V'} + \|\theta_m(t)\|^2_V + \|\theta_m(s)\|^2_V\right).$$

To estimate the integrals of the terms depending on t, it is enough to change the order of integration. Using the boundedness of the sequence θ_m in $L^2(0,T;V)$, we obtain the estimate of equicontinuity:

$$\int\limits_0^{T-\delta} \|\theta_m(s+\delta) - \theta_m(s)\|^2 ds \leq C\delta. \tag{13}$$

From the estimate (13) (passing, if necessary, to a subsequence), it follows that $\theta_m \longrightarrow \theta$ in $L^2(Q)$. Therefore, by virtue of the inequality $|h(\theta_m) - h(\theta)| \leq k_1|\theta_m - \theta|$, we conclude that $h(\theta_m) \longrightarrow h(\theta)$ in $L^2(Q)$ and hence $\chi = h(\theta)$. The theorem is proved.

Remark. If θ is an arbitrary weak solution of the problem (4)–(5), then it satisfies the estimates obtained in the proof of Theorem 1.

Indeed, by virtue of equality

$$\frac{1}{2}\frac{d}{dt}(\sigma\theta, \theta) = (\sigma\theta', \theta) + \frac{1}{2}(\sigma_t\theta, \theta),$$

where $\sigma_t = \partial\sigma/\partial t$, the function $t \longrightarrow d(\sigma\theta,\theta)/dt$ is integrable on $(0,T)$. Multiplying the first equation in (6), in the sense of the inner product of H, by θ and integrating with respect to time, as in the proof of Theorem 1, we derive

$$\|\theta(t)\|^2 \leq \frac{K}{\sigma_0}\exp\frac{\sigma_2 t}{\sigma_0} \quad \text{a.e. on } (0,T), \quad \nu\int\limits_0^T \|\theta(s)\|^2_V ds \leq K\exp\frac{\sigma_2 T}{\sigma_0}. \tag{14}$$

Let us show that the solution is unique in the class of functions with bounded gradient.

Theorem 2. *Let conditions (i)–(iv) hold. If θ_* is a weak solution to problem (4)–(5) such that $\nabla\theta_* \in L^\infty(Q)$, then there are no other solutions of the problem.*

Proof. Let θ_1 be another solution of the problem (4)–(5), $\theta = \theta_1 - \theta_*$. Then

$$\sigma\theta' + A(\theta_1) - A(\theta_*) = 0, \quad \theta(0) = 0.$$

We multiply the first equation, in the sense of the inner product of H, by θ and integrate with respect to time. Then

$$\frac{1}{2}\|\sqrt{\sigma}\theta\|^2 + \int_0^t \left(k(\theta_1)\nabla\theta, \nabla\theta) + \int_\Gamma \gamma\theta^2(s)d\Gamma \right) ds =$$

$$\frac{1}{2}\int_0^t (\sigma_t\theta, \theta)ds - \int_0^t ((k(\theta_1) - k(\theta_*))\nabla\theta_*, \nabla\theta)ds.$$

Using the constraints on the functions k, σ and their derivatives, we obtain the inequality

$$\frac{\sigma_0}{2}\|\theta\|^2 + k_0\int_0^t \|\nabla\theta\|^2 ds \leq \frac{\sigma_2}{2}\int_0^t \|\theta\|^2 ds + k_2\|\nabla\theta_*\|_{L^\infty(Q)}\int_0^t \|\theta\|\|\nabla\theta\|ds.$$

We take into account that $\|\theta\|\|\nabla\theta\| \leq \varepsilon\|\nabla\theta\|^2 + \dfrac{1}{4\varepsilon}\|\theta\|^2$ and estimate the last term assuming

$$\varepsilon = \frac{k_0}{k_2\|\nabla\theta_*\|_{L^\infty(Q)}}.$$

Then it follows from Gronwall's inequality that $\theta = 0$ and the solution θ_1 coincides with θ_*.

3.2. Initial-boundary value problem modeling EVLA

Let us reduce the problem (1)–(3) to the problem (4)–(5). For this purpose, we first consider the boundary value problem

$$-\operatorname{div}(\alpha\nabla\varphi) + \mu_a\varphi = \zeta, \quad x \in \Omega, \quad \alpha\partial_n\varphi + 0.5\varphi = 0 \text{ on } \Gamma. \tag{15}$$

Let the following conditions hold:

(r) $\quad 0 < \alpha_0 \leq \alpha(x) \leq \alpha_1, \quad 0 < \mu_0 \leq \mu_a(x) \leq \mu_1, \quad x \in \Omega.$

Let us define the operator $B : V \longrightarrow V'$,

$$(B\varphi, v) = (\alpha\nabla\varphi, \nabla v) + (\mu_a\varphi, v) + 0.5\int_\Gamma \varphi v d\Gamma \quad \forall v \in V.$$

It follows from the Lax-Milgram lemma that for any function $\zeta \in H$ there is a unique solution of the equation $B\varphi = \zeta$, which is a weak solution of the boundary value problem (15). Moreover, the inverse operator $B^{-1} : H \longrightarrow V$ is continuous.

The problem (1)–(3) is now reduced to problem (4)–(5) if we put

$$f = u_1\xi + u_2\mu_a B^{-1}\xi, \quad u_{1,2} \in \mathbb{R}.$$

Theorem 3. *Let conditions (i)–(iv), (r) hold. Then there exists a solution of the problem* (1)–(3), $\{\theta, \varphi\} \in L^2(0, T; V) \cap L^\infty(0, T; H) \times L^\infty(0, T; V)$. *If, in addition,* $\nabla\theta \in L^\infty(Q)$, *then the solution is unique.*

4. Numerical simulation

Let us present an iterative algorithm for solving the initial-boundary value problem (1)–(3). At each iteration of this algorithm, a linear initial-boundary value problem is solved:

$$\sigma\partial\theta^n/\partial t - \operatorname{div}(k(\theta^{n-1})\nabla\theta^n) = u_1\xi + u_2\mu_a B^{-1}\xi, \quad k(\theta^{n-1})\partial_n\theta^n + \gamma(\theta^n - \theta_b) = 0, \quad \theta^n|_{t=0} = \theta_0. \tag{16}$$

Here, θ^{n-1}, $n = 1, 2, ...$ is the temperature field obtained at the previous iteration. The solvability of the linear problem (16) is well known, and similarly to the proof of Theorem 1, the following estimates are derived:

$$\|\theta^n(t)\|^2 \le \frac{K}{\sigma_0}\exp\frac{\sigma_2 t}{\sigma_0} \quad \text{a.e. on } (0,T), \quad \nu\int_0^T \|\theta^n(s)\|_V^2 ds \le K\exp\frac{\sigma_2 T}{\sigma_0},$$

$$\int_0^{T-\delta} \|\theta^n(s+\delta) - \theta^n(s)\|^2 ds \le C\delta.$$

These estimates (passing, if necessary, to a subsequence) imply the convergence:

$$\theta^n \longrightarrow \theta \quad \text{weakly in } L^2(0,T;V), \quad \text{strongly in } L^2(0,T;H).$$

This allows us to go to the limit in (16), concluding that θ is a solution of the problem (1)–(3).

The efficiency of laser ablation can be estimated by the behavior of temperature profiles at different points of the computational domain [10, 11, 12]. The analysis of temperature profiles provides the possibility to estimate whether the temperature level inside the vein is sufficient for obliteration and, at the same time, whether the thermal effect outside the vein will be safe for living tissue. The calculations based on Eqs. (1)–(3) and for the computational domain presented in Fig. 1 are performed with the optical and thermophysical problem parameters presented in [10, 11]. The parameters θ_b and θ_0 are equal to 37, and $\gamma = 1$. In all the calculations, the initial position of the optical fiber tip corresponds to $z = 5$, and its motion velocity is 2 mm/s.

Figure 2 illustrates the behavior of temperature profiles corresponding to the wavelength of 810 nm at four observation points (1.5, 10) (inner vein wall), (2, 10), (2.5, 10) (outer vein wall), and (3.5, 10) (perivenous tissue). The source power is set as $(u_1, u_2) = (3, 7)$ (hereinafter, the power is given in watts). As can be seen from these plots, the perivenous tissue temperature remains quite safe despite a high temperature inside the vein.

Figure 3 shows the behavior of temperature profiles at the point (1.5, 10) for radiation with different wavelengths: 810 nm, 1064 nm, 1470 nm, and 1950 nm. The source power is set as $(u_1, u_2) = (3, 7)$ in all cases. As can be seen from the figure, a change of the radiation wavelength has a significant effect on the behavior of the temperature profile. It is possible to ensure the same boiling duration (when the temperature is more than 95°C) for temperature profiles corresponding to different wavelengths by changing the radiation power as it is shown in Fig. 4. At the same time, the temperature level in the perivenous tissue is quite safe. Here, in all cases, the ratio u_1/u_2 remains unchanged as 3/7.

As can be seen from the experiments, the use of computer simulation is a promising way to determine the optimal radiation parameters that ensure an efficient and safe procedure of EVLA.

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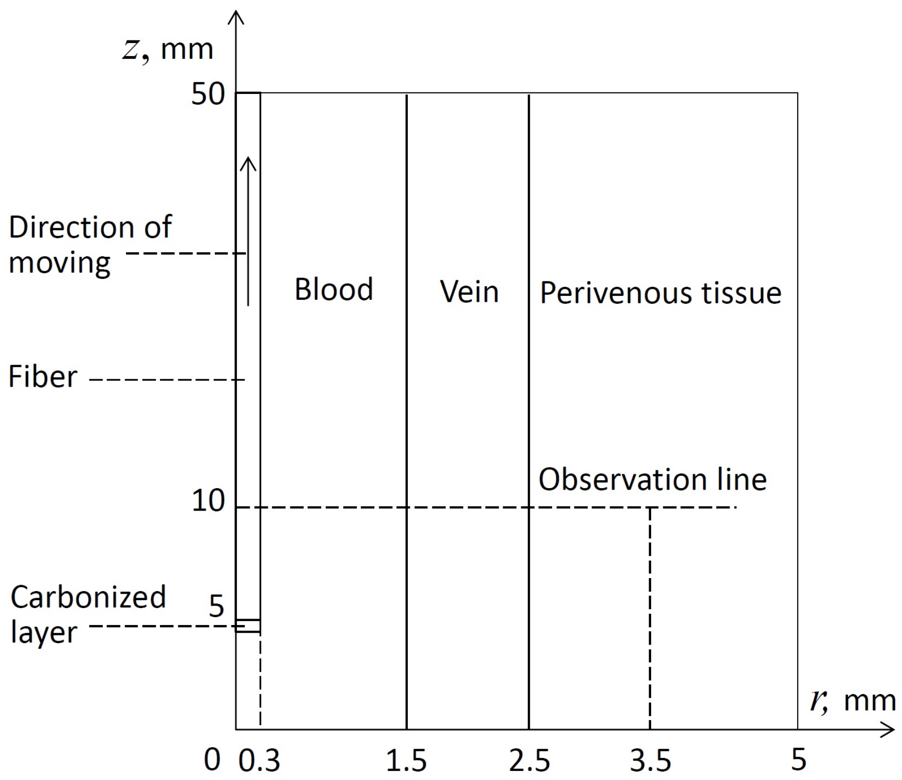

Figure 1: Computational domain.

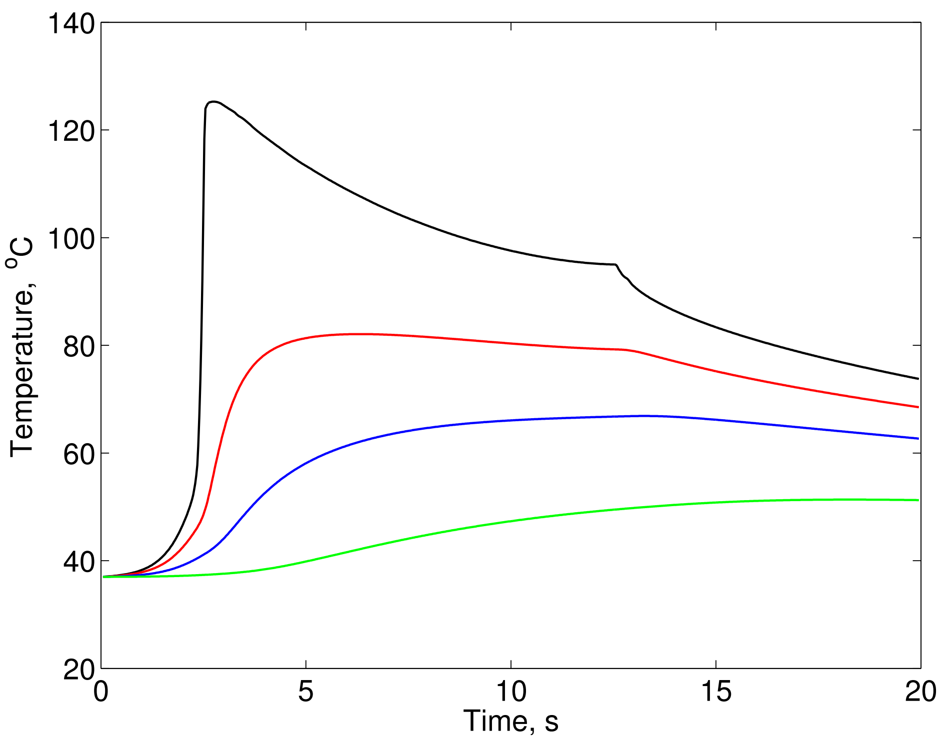

Figure 2: Temperature profiles at the points (1.5, 10) (black), (2, 10) (red), (2.5, 10) (blue), and (3.5, 10) (green).

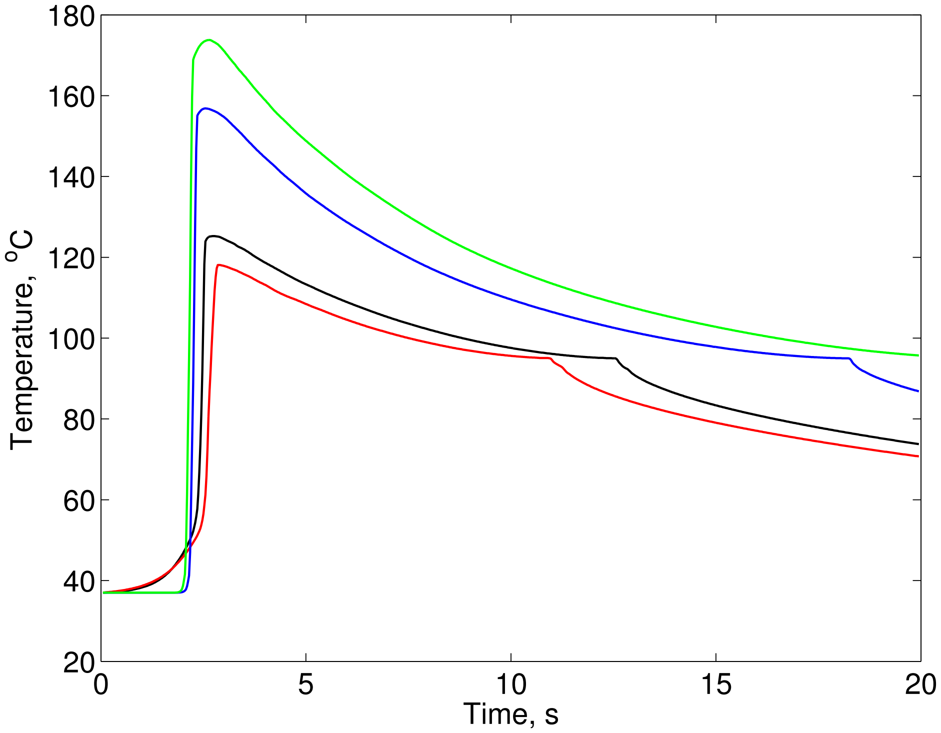

Figure 3: Temperature profiles at the points (1.5, 10) for laser power of 10 W and for different wavelengths: 810 nm (black), 1064 nm (red), 1470 nm (blue), and 1950 nm (green).

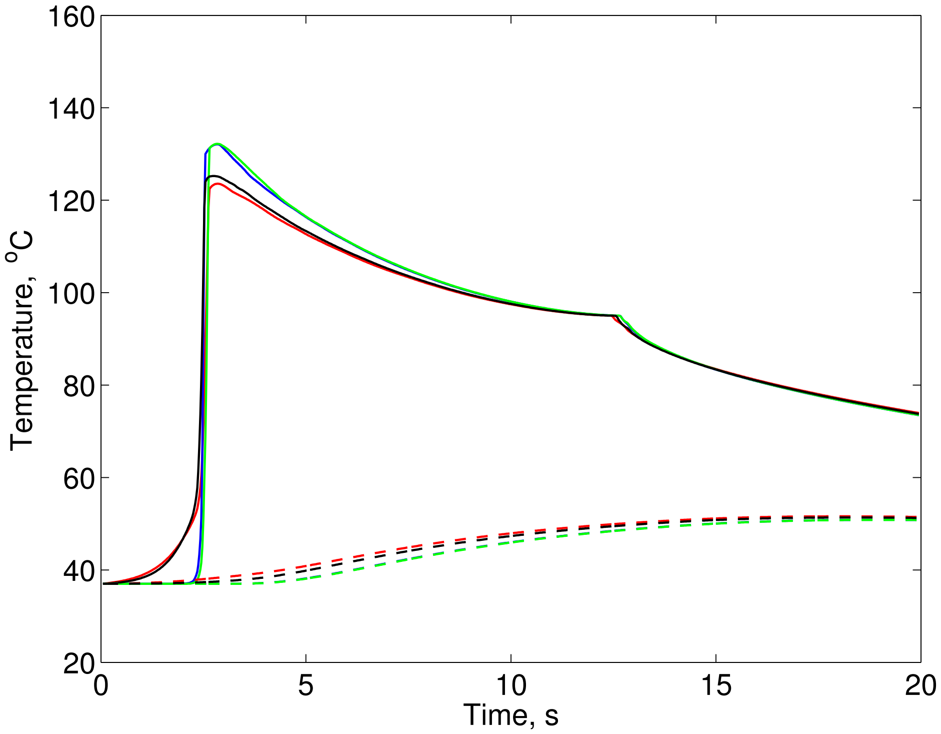

Figure 4: Temperature profiles at the points (1.5, 10) (solid) and (3.5, 10) (dashed) for different wavelengths and laser powers: 810 nm, 10 W (black), 1064 nm, 11 W (red), 1470 nm, 7.3 W (blue), and 1950 nm, 6.1 W (green).