On an explicit difference method for solving one nonlinear parabolic equation with double degeneration and nonlocal spatial operator.

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Abstract

We consider the initial-boundary value problem for nonlinear parabolic equation. This type of equation can be classified as a parabolic equation with double degeneration: degeneration can be present in space operator, and a nonlinear function which is under the derivative sign with respect to the variable t, may not be separated from zero. The space operator of the considered equation nonlinearly depends on the sought function, its gradient and the non-local (integral) solution characteristic. This problem has an applied nature. Such equations appear, for example, in modeling the process of bacteria population spreading. In the present paper we propose and investigate the explicit differential scheme. A priori estimates are obtained, and the convergence of constructed algorithm is proved. The current work is a continuation of the research begun in the works [1], [2], [3], where the existence and uniqueness theorems for the generalized solution have been proved, the convergence of the finite-element method scheme and the explicit difference scheme in the case when nonlinearity is present only in the spatial operator have been investigated. In paper [4] for a problem with double degeneration, an approximate method has been studied. That method was constructed with the use of semidiscretization with respect to a variable t and the finite element method in the space variable with lowering nonlocality to the lower layer, the existence of an approximate solution and the convergence of the constructed algorithms were proved.

Keywords

parabolic equation, nonlocal spatial operator, double degeneration, convergence

1. Statement of the problem

Let the Ω be bounded domain in the space R^n, Γ is its boundary, Ω, $Q_T = \Omega \times (0, T)$. In the domain Q_T consider the initial-boundary value problem

$$\frac{\partial \varphi(u)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(k_i(x, u, \nabla u, Bu)\big) = f, \quad x \in \Omega, \ t \in (0, T), \tag{1}$$

$$u(x, 0) = u_0(x) \quad x \in \Omega, \qquad u(x, t) = 0, \quad x \in \Gamma, \quad t \in [0, T]. \tag{2}$$

Here k_i, u_0 are known functions, B is an operator of the form

$$Bu(t) = \int\limits_{\Omega'} g(x, u(x, t))\, dx\,, \tag{3}$$

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g is a given function, Ω' is a domain that is contained in Ω or coincides with it.

Lets define the operator L

$$Lu = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(k_i(x, u, \nabla u, Bu)\big).$$

We assume that function $\varphi(\xi)$ is an absolutely continuous, strongly increasing function and it satisfy the following inequalities for arbitrary $\xi \in R^1$,

$$b_0 \mid \xi \mid^{\alpha} - b_1 \leq \Phi(\xi) \equiv \int_0^{\xi} \varphi'(t)t dt \leq b_2 \mid \xi \mid^{\alpha} + b_3, \quad \alpha > 1, \tag{4}$$

$$\mid \varphi(\xi) \mid \leq b_i \mid \xi \mid^{\alpha-1} + b_5, \tag{5}$$

$$(\varphi'(\xi)\xi)' \geq 0, \tag{6}$$

here b_{ij} are constants such that following inequalities are correct

$$b_{0i} > 0, \;\; b_{1i} \geq 0, \;\; b_{2i} > 0, \;\; b_{3i} \geq 0, \;\; b_{4i} > 0, \;\; b_{5i} \geq 0\,, i = 1, 2,$$

functions $k_i(x, \xi_0, \xi, v)$, $i = 1, \ldots, n$, are continuous with respect to ξ_0, v and ξ, measurable with respect to x and for arbitrary $x \in \Omega$, $\xi_0, v \in R$, $\xi^1, \xi^2, \xi \in R^n$ satisfy the following conditions

$$\mid k_i(x, \xi_0, \xi, v) \mid \leq d_0 \sum_{j=1}^{n} \mid \xi_j \mid^{p-1} + d_1\,, \quad d_0 > 0, \quad d_1 \geq 0\,, \quad p > 1\,, \tag{7}$$

$$\sum_{i=1}^{n} k_i(x, \xi_0, \xi, v)\xi_i \geq d_2 \sum_{i=1}^{n} \mid \xi_i \mid^{p} - d_3, \quad d_2 > 0, \quad d_3 \geq 0, \tag{8}$$

$$\sum_{i=1}^{n} \big(k_i(x, \xi_0, \xi^1, v) - k_i(x, \xi_0, \xi^2, v)\big)(\xi_i^1 - \xi_i^2) \geq 0. \tag{9}$$

Lets note that the condition (7) implies that the operator L, acting from $\overset{\circ}{W}{}_p^1(\Omega)$ into $W_{p'}^{-1}(\Omega)$, where $p' = \dfrac{p}{p-1}$, is bounded. The conditions (8), (9) provide, respectively, the coercivity and monotonicity with respect to the gradient of the operator L.

We assume that the function $g(x, \xi)$, defining the operator B, is continuous with respect to ξ, measurable with respect to x and satisfies the following condition

$$|g(x, \xi)| \leq g_0(x) + |\xi|^s \qquad \text{for almost all } \; x \in \Omega, \tag{10}$$

where g_0 is a function integrable over Ω, $s \geq 0$.

Space operators with non-localities of the form (3) arise, for example, in the mathematical describing the diffusion of bacteria population when it is assumed that the propagation speed at a point is specified by the global state of environment (e.g., see [5], [6]).

Lets define a generalized solution for a problem (1)–(2).

A function $u \in L_p(0, T; \overset{\circ}{W}{}_p^1(\Omega)) \bigcap L_\infty(0, T; L_\alpha(\Omega))$ such that

$$u(x, 0) = u_0(x) \;\; \text{almost everywhere in } \Omega, \qquad \frac{\partial \varphi(u)}{\partial t} \in L_{p'}(0, T; W_{p'}^{-1}(\Omega)), \tag{11}$$

will be called a generalized solution of problem (1), (2), if for any function v from $L_p(0,T;\overset{\circ}{W}{}^1_p(\Omega))$ the integral identity holds

$$\int_0^T \left\langle \frac{\partial \varphi(u)}{\partial t}, v \right\rangle dt + \int_0^T \int_\Omega \sum_{i=1}^n k_i\big(x,u,\nabla u, Bu\big)\,\frac{\partial v}{\partial x_i}\,dxdt = \int_0^T \langle f, v\rangle dt, \tag{12}$$

here $\langle g, v\rangle$ is the value of a functional g from $W^{-1}_{p'}(\Omega)$ on element v from $\overset{\circ}{W}{}^1_p(\Omega)$.

2. Auxiliary results and notation

In what follows, we will assume that the domain Ω is a n-dimensional parallelepiped: $\overline{\Omega} = \big\{x \in R_n : 0 \le x_i \le l_i,\ i = 1,2,\dots,n.\big\}$. On Ω construct a uniform mesh $\bar\omega_h$ with a mesh step h_i in the i-th direction, $\vec{h} = (h_1,\dots,h_n)$, $h = \min\limits_{1\le i\le n} h_i$. We will assume that there is a constant c such that $\overline{h} \le ch$, $\overline{h} = \max\limits_{1\le i\le n} h_i$. We denote

$$\overline{\omega}_h = \left\{x = (x_1,\dots,x_n) \in \overline{\Omega}\ :\ x_i = jh_i,\ j = 0,\dots,N_i,\ N_i = \frac{l_i}{h_i}\right\},$$

$$\gamma_h = \bar\omega_h \cap \Gamma, \quad \omega_h = \overline{\omega}_h \backslash \gamma_h.$$

On $[0,T]$ we construct a uniform mesh with a step τ:

$$\overline{\omega}_\tau = \left\{t \in [0,T]\ :\ t = j\tau,\ j = 0,\dots,M,\ M = \frac{T}{\tau}\right\}, \quad \omega_\tau = \overline{\omega}_\tau \backslash \{0\}.$$

We denote by H the set of mesh functions defined on $\overline{\omega}$, $\overset{\circ}{H}$ are the functions from H, that equal zero on γ. Let further r is the n-dimensional vector with coordinates

$$r_i = \pm 1,\ \ \nabla_r y(x) = (\partial_{r_1} y(x), \partial_{r_2} y(x), \dots, \partial_{r_n} y(x)),$$

$$\partial_{r_i} y(x) = \begin{cases} y_{x_i}(x), & r_i = +1, \\ y_{\bar{x}_i}(x), & r_i = -1. \end{cases}$$

Let us denote by $H_r(x)$ a mesh cell $\overline{\omega}$, , which contains all the mesh points participating in the notation of operator $\nabla_r y(x)$, ω_r is the set of points $x \in \overline{\omega}$, at which the operator $\nabla_r y(x)$ is defined. In the space of mesh functions $\overset{\circ}{H}$ introduce the following norms and scalar products

$$(y,v)_r = \sum_{x\in\omega_r} \tilde{H}_r\, y(x)\, v(x), \qquad [y,v] = (1/2^n)\sum_r (y,v)_r,$$

$$\| y \|_p = [|\, y\,|^p, 1]^{1/p}, \qquad \| y \|^p_{+p} = (1/2^n)\sum_r \sum_{i=1}^n (|\,\partial_{r_i} y\,|^p, 1)_r,$$

$$\| y \|_{-p'} = \sup_{v\ne 0} \frac{[y,v]}{\| v \|_{+p}},$$

here $\tilde{H}_r = \text{mes}\, H_r(x)$.

For mesh functions, we define piecewise constant extensions x and t each

$$\Pi_r z(x) \quad = \quad \{z(x'), x' \in \omega_r, x \in H_r(x')\},$$

$$\begin{aligned}\Pi^- w(t') &= \{w(t), t = k\tau, (k-1)\tau < t' \leq k\tau\},\\ \Pi^+ w(t') &= \{w(t), t = k\tau, k\tau \leq t' < (k+1)\tau\},\\ \Pi_r^+ w &= \Pi^+\Pi_r w, \quad \Pi_r^- w = \Pi^-\Pi_r w.\end{aligned}$$

Lemma 1. (See [7]) If $\varphi(\xi)$ is an absolutely continuous increasing function, then the following inequality holds

$$(\varphi(\xi) - \varphi(\eta))\xi \geq \Phi(\xi) - \Phi(\eta), \quad \forall \xi, \eta \in R^1. \tag{13}$$

Lemma 2. (See [7]) Let $\alpha \geq 2$, function φ satisfies the condition (4) and besides

$$\varphi'(\xi) \geq b_6 \mid \xi \mid^{\alpha-2}, \qquad b_6 > 0. \tag{14}$$

Then for any constant $\theta > 1$ there is $\bar{c} = const > 0$, such that for any $\xi, \eta \in R^1$ the inequality holds

$$(\varphi(\xi) - \varphi(\eta))(\theta\xi - (\theta-1)\eta) \geq \Phi(\xi) - \Phi(\eta) + \bar{c} \mid \xi - \eta \mid^{\alpha}. \tag{15}$$

Lemma 3. (See [7]) Let $\varphi(\xi)$ be an absolutely continuous, monotonically increasing function satisfying the conditions(4)–(6). Then for any function v such that

$$v \in L_p(0, T; \overset{\circ}{W}{}_p^1 (\Omega)) \bigcap L_\infty(0, T; L_\alpha(\Omega)), \tag{16}$$

$$\frac{\partial \varphi(v)}{\partial t} \in L_{p'}(0, T; W_{p'}^{-1}(\Omega)), \tag{17}$$

$$v(x, 0) \in \overset{\circ}{W}{}_p^1 (\Omega) \bigcap L_\alpha(\Omega), \tag{18}$$

the following equality holds

$$\int_0^T \langle \frac{\partial \varphi(v)}{\partial t}, v\rangle \, dt = \lim_{\lambda \to 0} \frac{1}{\lambda} \int_{T-\lambda}^T \int_\Omega \Phi(v(t)) \, dx \, dt - \int_\Omega \Phi(v(0)) \, dx. \tag{19}$$

It is easy to check the validity of the following lemma.

Lemma 4. (See [7]) For any $y \in \overset{\circ}{H}$ the inequality holds

$$\| y \|_{+p} \leq \lambda_\alpha \| y \|_\alpha, \tag{20}$$

where $\lambda_\alpha = \dfrac{c\sqrt[p]{n}}{h^{1+n(p-\alpha)/\alpha p}}$, if $p \geq \alpha$ and $\lambda_\alpha = \dfrac{c\sqrt[p]{n}}{h}$, if $1 < p < \alpha$.

3. Construction and investigation of an explicit difference scheme

For the problem (1), (2), consider the explicit difference scheme

$$\varphi_t(y) + Ay(x, t) = f_{h\tau}(x, t), \quad x \in \omega_h, \; t \in \overline{\omega}_\tau \backslash \{T\}, \tag{21}$$

$$y(x, 0) = y_0(x), \qquad y \mid_{\gamma_h} = 0.$$

Here A is a difference operator acting from $\overset{\circ}{H}$ to $\overset{\circ}{H}$, defined by the relation

$$[Ay, w] = \frac{1}{2^n} \sum_r \sum_{i=1}^n (a_i(x, y) k_i(x, \nabla_r y, B_h y), \partial_{r\,i} w)_r,$$

where $B_h y(t) = B(2^{-n} \sum_r \Pi_r y(t))$, y_0 a difference analog of u_0 such that

$$\Pi_r y_0 \longrightarrow u_0 \quad \text{in} \quad L_\alpha(\Omega), \tag{22}$$

$f_{h\tau}$ is a mesh function, that is an approximation of the original equation right side, which we define as follows

$$[f_{h\tau}, v] = \frac{1}{2^n} \sum_r \sum_{i=0}^{n} (f^r_{h\tau,i}, \partial_{r_i} v)_r \quad \forall v \in \overset{\circ}{H},$$

where

$$\partial_{r_0} v \equiv v, \quad f^r_{h\tau,i}(t) = \frac{1}{\tau \ \operatorname{mes}\big(H_r(x)\big)} \int_t^{t+\tau} \int_{H_r(x)} f_i(\xi, \eta)\, d\xi d\eta.$$

Conditions (7)–(8) on the coefficients k_i provide continuity, boundedness:

$$\| Ay \|_{-p'} \le c_0 \| y \|^{p-1}_{+p} + \bar{c}_0, \tag{23}$$

coercivity of the operator A :

$$[Ay, y] \ge d_2 \| y \|^p_{+p} - d_3, \tag{24}$$

with constants $d_2 > 0$, $d_3 \ge 0$, $c_0 > 0$, $\bar{c}_0 \ge 0$, independent on $\bar{h}$ and τ. The unique solvability of the difference scheme (21) follows from the condition that the function φ is strictly monotonic.

Lemma 5. Let $\alpha \ge 2$, function φ satisfies the conditions (4)–(5) and besides

$$u_0 \in L_\alpha(\Omega), \quad f \in L_q(0, T; W^{-1}_{p'}(\Omega)), \ q = \max\{\alpha', p'\}.$$

Then for any

$$\tau \le \begin{cases} c\, \dfrac{h^\alpha}{2^\alpha n^{\alpha/p}}, & 1 < p < \alpha, \\[2ex] c\, \dfrac{h^{p+n(p-\alpha)/\alpha}}{2^p n}, & p \ge \alpha, \end{cases} \tag{25}$$

for the solution of the difference scheme (21) the following a priori estimates hold

$$\sum_{t=0}^{t'} \tau \| y \|^p_{+p} \le const, \tag{26}$$

$$\max_{t' \in \bar{\omega}_\tau} \| y(t') \|^\alpha_\alpha \le const, \tag{27}$$

$$\sum_{t=0}^{t'} \tau^\alpha \| y_t \|^\alpha_\alpha \le const \qquad \forall t' \in \bar{\omega}_\tau, \tag{28}$$

$$\frac{1}{k\tau} \sum_{t=0}^{T-k\tau} \tau[\varphi(y(t + k\tau)) - \varphi(y(t)), y(t + k\tau) - y(t)] \le const \tag{29}$$

$$\forall k \in \{1, \dots, N\}.$$

Proof. Multiply both sides of (21) scalarly in H by $\tau(\theta\hat{y} - (\theta - 1)y)$, where the constant $\theta > 1$. As a result, we get

$$\tau[\varphi_t(y), \theta\hat{y} - (\theta - 1)y] + \tau[Ay, \theta\hat{y} - (\theta - 1)y] = \tau[f_{h\tau}, \theta\hat{y} - (\theta - 1)y]$$

or

$$\tau[\varphi_t(y), \theta\hat{y} - (\theta - 1)y] + \tau[Ay, y] = \tau[f_{h\tau}, y] + \tau^2\theta[f_{h\tau}, y_t] - \tau^2\theta[Ay, y_t]. \tag{30}$$

Using lemma 2, we estimate the first summand in the left-hand side of the equation (30)

$$\tau[\varphi_t(y), \theta\hat{y} - (\theta - 1)y] \geq [\Phi(\hat{y}) - \Phi(y), 1] + \bar{c}\tau^\alpha \parallel y_t \parallel_\alpha^\alpha . \tag{31}$$

To estimate the first two summands on the right-hand side of (30) we use Hölder inequality, ε − inequality and a difference analogue of the Friedrichs inequality, as a result we have

$$\begin{aligned} \tau[f_{h\tau}, y] &\leq \frac{1}{\varepsilon_1^{p'} p'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j} \parallel_{p'}^{p'} + \frac{\varepsilon_1^p}{p}(1 + c_\Omega)\tau \parallel y \parallel_{+p}^p, & (32)\\ \tau^2[f_{h\tau}, y_t] &\leq \frac{1}{\varepsilon_2^{\alpha'}\alpha'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j} \parallel_{p'}^{\alpha'} + \frac{\varepsilon_2^\alpha \tau^{\alpha+1}}{\alpha}(\parallel y_t \parallel_{+p}^\alpha + \parallel y_t \parallel_p^\alpha) \leq & (33)\\ &\leq \frac{1}{\varepsilon_2^{\alpha'}\alpha'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j} \parallel_{p'}^{\alpha'} + \frac{\varepsilon_2^\alpha \tau^{\alpha+1}}{\alpha}(1 + c_\Omega)\lambda_\alpha^\alpha \parallel y_t \parallel_\alpha^\alpha + c_1\tau, \end{aligned}$$

here c_Ω is the constant from the difference analog of the Friedrichs inequality. From (23) follows that

$$\tau^2\theta[Ay, y_t] \leq \tau^2\theta(c_0 \parallel y \parallel_{+p}^{p-1} + \bar{c}_0) \parallel y_t \parallel_{+p} \equiv I + \tau^2\theta\bar{c}_0 \parallel y_t \parallel_{+p} . \tag{34}$$

Further, using (31)–(34) and the coercivity of the operator A, from (30) is easy to obtain

$$\begin{aligned} &[\Phi(\hat{y}) - \Phi(y), 1] + \bar{c}\tau^\alpha \parallel y_t \parallel_\alpha^\alpha + d_2\tau \parallel y \parallel_{+p}^p - d_3\tau \leq \\ &\leq \frac{1}{\varepsilon_1^{p'} p'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j} \parallel_{p'}^{p'} + \frac{\varepsilon_1^p}{p}(1 + c_\Omega)\tau \parallel y \parallel_{+p}^p + \\ &+ \frac{1}{\varepsilon_2^{\alpha'}\alpha'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j} \parallel_{p'}^{\alpha'} + \frac{\varepsilon_2^\alpha \tau^{\alpha+1}}{\alpha}(2 + c_\Omega)\lambda_\alpha^\alpha \parallel y_t \parallel_\alpha^\alpha + I + c_1\tau. \end{aligned} \tag{35}$$

Let $p \geq \alpha$. We estimate I using Hölder's inequality and lemma 2, as a result we obtain

$$\begin{aligned} I &\leq \tau^2 c_0\theta \parallel y \parallel_{+p}^{p/\alpha'} \parallel y \parallel_{+p}^{(p-\alpha)/\alpha} \lambda_\alpha \parallel y_t \parallel_\alpha \leq \tau^2 c_0\theta \parallel y \parallel_{+p}^{p/\alpha'} \lambda_\alpha^{p/\alpha} \parallel y \parallel_\alpha^{(p-\alpha)/\alpha} \parallel y_t \parallel_\alpha \leq \\ &\leq \frac{\tau\varepsilon_3^{\alpha'}}{\alpha'} \parallel y \parallel_{+p}^p + \frac{\tau^{\alpha+1}(c_0\theta)^\alpha\lambda_\alpha^p}{\alpha\varepsilon_3^\alpha} \parallel y \parallel_\alpha^{p-\alpha} \parallel y_t \parallel_\alpha^\alpha . \end{aligned} \tag{36}$$

Substituting (36) into (35) and summing the resulting inequalities over t from 0 to $t' \in \bar{\omega}_\tau$, we will have

$$\begin{aligned} &[\Phi(y(t')), 1] + \left(M_2 - \frac{\varepsilon_1^p}{p}(1 + _\Omega^p) - \frac{\varepsilon_3^{\alpha'}}{\alpha'}\right)\sum_{t=0}^{t'} \tau \parallel y \parallel_{+p}^p + \\ &+ \sum_{t=0}^{t'}\left(\bar{c} - \tau\frac{\varepsilon_2^\alpha}{\alpha}(2 + c_\Omega)\lambda_\alpha^\alpha - (c_0\theta)^\alpha\frac{\tau\lambda_\alpha^p}{\alpha\varepsilon_3^p} \parallel y(t) \parallel_\alpha^{p-\alpha}\right)\tau^\alpha \parallel y_t \parallel_\alpha^\alpha \leq \\ &\leq \frac{1}{\varepsilon_1^{p'} p'}\sum_{t=0}^{t'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j}(t) \parallel_{p'}^{p'} + \frac{1}{\varepsilon_2^{\alpha'}\alpha'}\sum_{t=0}^{t'}\tau\sum_{j=0}^{n} \parallel f_{h\tau,j}(t) \parallel_{p'}^{\alpha'} + [\Phi(y(0)), 1] + c_3. \end{aligned} \tag{37}$$

First, let us prove that (37) implies the estimate

$$\begin{aligned}\| y(t') \|_\alpha^\alpha \quad \le \quad & c\Bigg(\sum_{t=0}^{T} \tau \sum_{j=0}^{n} \| f_{h\tau,j}(t) \|_{p'}^{p'} + \sum_{t=0}^{T} \tau \sum_{j=0}^{n} \| f_{h\tau,j}(t) \|_{p'}^{\alpha'} + \\ & +[\Phi(y(0)), 1] + 1\Bigg) = m^\alpha \qquad \forall t' \in \bar{\omega}_\tau, \end{aligned} \tag{38}$$

where c, m are constants independent of $\bar{h}$ and τ. For $t' = 0$ estimate (38) holds. We assume that (38) is valid for all $t' \le t_1;\ t', t_1 \in \omega_\tau$. Let us prove that (38) holds for $t' = t_1 + \tau$. To do this, write inequality (37) for $t' = t_1 + \tau$, considering, that $\| y(t) \|_\alpha^\alpha \le m^\alpha \quad \forall t \le t_1$,

$$\begin{aligned}&[\Phi(y(t_1 + \tau)), 1] + \left(d_2 - \frac{\varepsilon_1^p}{p}(1 + c_\Omega^p) - \frac{\varepsilon_3^{\alpha'}}{\alpha'} \right) \sum_{t=0}^{t_1} \tau \| y \|_{+p}^p + \\ &+\left(\bar{c} - \tau \frac{\varepsilon_2^\alpha}{\alpha}(2 + c_\Omega)\lambda_\alpha^\alpha - (c_0\theta)^\alpha \frac{\tau\lambda_\alpha^p}{\alpha\varepsilon_3^p} m^{p-\alpha} \right) \sum_{t=0}^{t_1} \tau^\alpha \| y_t \|_\alpha^\alpha \le \\ &\le \frac{1}{\varepsilon_1^{p'} p'} \sum_{t=0}^{t_1} \tau \sum_{j=0}^{n} \| f_{h\tau,j}(t) \|_{p'}^{p'} + \frac{1}{\varepsilon_2^{\alpha'} \alpha'} \sum_{t=0}^{t_1} \tau \sum_{j=0}^{n} \| f_{h\tau,j}(t) \|_{p'}^{\alpha'} +[\Phi(y(0)), 1] + c_3. \end{aligned} \tag{39}$$

Choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3, \bar{h}$ and τ so that

$$\begin{aligned} d_2 - \frac{\varepsilon_1^p}{p}(1 + c_\Omega^p) - \frac{\varepsilon_3^{\alpha'}}{\alpha'} \quad &\ge \quad \delta_1 > 0, \\ \bar{c} - \tau \frac{\varepsilon_2^\alpha}{\alpha}(2 + c_\Omega)\lambda_\alpha^\alpha - (c_0\theta)^\alpha \frac{\tau\lambda_\alpha^p}{\alpha\varepsilon_3^p} m^{p-\alpha} \quad &\ge \quad \delta_2 > 0\ , \end{aligned} \tag{40}$$

and using the condition (4), of (39) is easy to obtain (38) for $t' = t_1 + \tau$. Therefore, the estimate (38) will be valid for any $t' \in \bar{\omega}_\tau$. From (37) and (38) the estimates (26)–(28) follow. Note that the constant c in (25) is chosen so that the inequality (40) holds.

Similarly to the way above, it is easy to verify the validity of estimates(26)–(28) in the case $1 < p < \alpha$.

Let us further prove the validity of the estimate (29).To do this, we sum both sides (21) over t from $\bar{t}$ to $\bar{t} + (k-1)\tau$, then multiply the resulting equality scalarly in H by $\tau(y(\bar{t} + k\tau) - y(\bar{t}))$ and again sum over $\bar{t}$ from 0 to $T - k\tau$, as a result we will have

$$\begin{aligned}&\frac{1}{k\tau} \sum_{\bar{t}=0}^{T-k\tau} \tau[\varphi(y(\bar{t} + k\tau)) - \varphi(y(\bar{t})), y(\bar{t} + k\tau) - y(\bar{t})] = \\ &= -\frac{1}{k} \sum_{\bar{t}=0}^{T-k\tau} \sum_{t=\bar{t}}^{\bar{t}+(k-1)\tau} \tau[Ay(t), y(\bar{t} + k\tau) - y(\bar{t})] + \frac{1}{k} \sum_{\bar{t}=0}^{T-k\tau} \sum_{t=\bar{t}}^{\bar{t}+(k-1)\tau} \tau[f, y(\bar{t} + k\tau) - y(\bar{t})]. \end{aligned} \tag{41}$$

Using the boundedness property of the operator A, Hölder's inequalities and (34), from (41) it is easy to obtain

$$\frac{1}{k\tau} \sum_{\bar{t}=0}^{T-k\tau} \tau[\varphi(y(\bar{t} + k\tau)) - \varphi(y(\bar{t})), y(\bar{t} + k\tau) - y(\bar{t})] \le c_1 \sum_{\bar{t}=0}^{T-k\tau} \tau \| y(\bar{t}) \|_{+p}^p + \frac{2}{p'} \sum_{t=0}^{T} \tau \sum_{j=0}^{n} \| f_{h\tau,j}(t) \|_{p'}^{p'}\ .$$

From the last inequality and (26) it follows (29). The lemma is proved.

The a priori estimates (26), (27) imply the boundedness of the set $\{\Pi_r^{\pm} y\}$ in the spaces $L_p(Q_T)$ and $L_\infty(0,T;L_2(\Omega)$, as well as the boundedness of the set $\{\Pi_r^{\pm}\partial_{r_i} y\}$ in the space $L_p(Q_T)$. Due to the weak compactness of bounded sets in reflexive spaces and the *-weak compactness of bounded sets in $L_\infty(0,T;L_\alpha(\Omega)$ there exists subsequences $\left\{\vec{h}^{(m)}\right\}_{m=1}^{\infty}$, $\left\{\tau_m\right\}_{m=1}^{\infty}$ [1] and the element u, which belongs to $L_p(0,T;\overset{\circ}{W}{}_p^1(\Omega))\bigcap L_\infty(0,T;L_2(\Omega)$, such that for $\vec{h}^{(m)}, \tau_m \longrightarrow 0$

$$\Pi_r^{\pm} y \rightharpoonup u \ \text{ in } \ L_p(Q_T), \tag{42}$$

$$\Pi_r^{\pm}\partial_{r_i} y \rightharpoonup \frac{\partial u}{\partial x_i} \ \text{ in } \ L_p(Q_T), \tag{43}$$

$$\Pi_r^{\pm} y \longrightarrow u \ \text{ *-weak in } \ L_\infty(0,T;L_\alpha(\Omega). \tag{44}$$

Using the estimates (27), (28), (30) and the mesh analogue of the compactness theorem (see [7], lemma 9), it is easy to confirm the existence of subsequences $\left\{\vec{h}^{(m)}\right\}_{m=1}^{\infty}$, $\left\{\tau_m\right\}_{m=1}^{\infty}$, for which, along with (42)– (44) the limit relation of the form below holds

$$\Pi_r^{\pm} y \longrightarrow u \ \text{ almost everywhere in } \ Q_T. \tag{45}$$

Further, the condition (7) and the estimate (26) imply the boundedness in the space $L_{p'}(Q_T)$ of the set $\left\{\Pi_r^{\pm} k_i(x,y,\nabla_r y, B_h y)\right\}$ for any $i\in\{1,2,\dots,n\}$. Therefore, there are $K_i \in L_{p'}(Q_T)$ and sequences $\left\{\vec{h}^{(m)}\right\}_{m=1}^{\infty}$, $\left\{\tau_m\right\}_{m=1}^{\infty}$ such that

$$\Pi_r^{\pm} k_i(x,y,\nabla_r y, B_h y) \rightharpoonup K_i \ \text{ in } \ L_{p'}(Q_T). \tag{46}$$

For $s\le\alpha$ from (27), (45) and Lebesgue's theorem on passage to the limit, it is easy to show that

$$\Pi^{\pm} B(y) \longrightarrow Bu \qquad \text{in } \ L_1(0,T). \tag{47}$$

Theorem 1. Let the functions φ, k_i satisfy conditions (7)–(9), (14), $\alpha \ge 2$ and the inequality (25) holds. Let, in addition, for $\tau, \bar{h} \longrightarrow 0$

$$\tau\lambda_\alpha^p \longrightarrow 0, \quad \text{if} \quad p\ge\alpha, \qquad \tau\lambda_\alpha^\alpha \longrightarrow 0, \quad \text{if} \quad 1<p<\alpha. \tag{48}$$

Then for any function $f\in L_q(0,T;W_{p'}^{-1}(\Omega))$, where $q=\max\{\alpha',p'\}$, and $u_0,\in L_\alpha(\Omega)\bigcap \overset{\circ}{W}{}_p^1(\Omega)$ subsequence of piecewise constant extensions of the solution to the difference scheme (21), defined by the relations (42)–(47), converges to a generalized solution of the problem (1)–(2).

Proof of this theorem is close to the proof of Lemma 3 from ([3]). Therefore, we present here only fragments of reasoning different from Lemma 3.

Let's scalarly multiply the difference scheme (21) by τz, where z – drift of the function $\bar{z}$ from $C^\infty(0,T;C_0^\infty(\Omega))$, $\bar{z}(x,T)=0$ and sum over t from 0 to $T-\tau$. As a result we get

$$\sum_{t=0}^{T-\tau}\tau[\varphi_t,z] + \sum_{t=0}^{T-\tau}\tau[Ay,z] = \sum_{t=0}^{T-\tau}\tau[f_{h\tau},z].$$

We transform the first summand by using the formula for summation by parts. We write the resulting equality using piecewise constant extensions in the form of the integral identity

$$\frac{1}{2^n}\sum_r\left\{-\int_0^T\int_\Omega \Pi_r^-\varphi(y)\Pi_r^-(z_{\bar{t}})dxdt + \sum_{i=1}^n\int_0^T\int_\Omega \Pi_r^+ k_i(x,y,\nabla_r y,B_h y)\Pi_r^+\partial_{r_i} z dxdt\right\} =$$

[1] In what follows, for the selected subsequences we will keep the notation of the sequences themselves.

$$= \frac{1}{2^n}\sum_r \sum_{i=1}^n \int_0^T \int_\Omega \Pi_r^+ f_{h\tau,i} \Pi_r^+ \partial_{r_i} z\,dx\,dt. \quad (49)$$

In the equality (49) , we pass to the limit as $\tau, h \longrightarrow 0$. As a result, we will have

$$-\int_0^T \int_\Omega \varphi(u)\frac{\partial \bar{z}}{\partial t}dx\,dt - \int_\Omega \varphi(u_0)\bar{z}(x,0)dx + \sum_{i=1}^n \int_0^T \int_\Omega K_i \frac{\partial \bar{z}}{\partial x_i}dx\,dt = \int_0^T \langle f, \bar{z}\rangle dt. \quad (50)$$

Following ([3], lemma 3), from (50) it is easy to obtain that

$$\int_0^T \langle \frac{\partial \varphi(u)}{\partial t}, \bar{z}\rangle\, dt + \sum_{i=1}^n \int_0^T \int_\Omega K_i \frac{\partial \bar{z}}{\partial x_i}dx\,dt = \int_0^T \langle f, \bar{z}\rangle\, dt \qquad \forall \bar{z} \in L_p(0,T; \overset{\circ}{W}{}_p^1(\Omega)) \quad (51)$$

and, besides, $u(x,0) = u_0(x)$ almost everywhere in Ω. Let us prove further that

$$\sum_{i=1}^n \int_0^T \int_\Omega K_i \frac{\partial \bar{z}}{\partial x_i}dx\,dt = \sum_{i=1}^n \int_0^T \int_\Omega k_i(x,u,\nabla u, Bu)\frac{\partial \bar{z}}{\partial x_i}dx\,dt \quad (52)$$

for any function $\bar{z}$ from $L_p(0,T; \overset{\circ}{W}{}_p^1(\Omega))$. To do this, we consider the following inequality

$$[\varphi(\hat{y}) - \varphi(y), \hat{y}] + \sum_{i=1}^n \tau[(k_i(x,y,\nabla y, B_h y) - k_i(x, \nabla\hat{v}, B_h y)), \partial_{r_i}(y - \hat{v})] \ge [\Phi(\hat{y}) - \Phi(y), 1], \quad (53)$$

where function y is the solution of the difference scheme (21), $v(x,t)$ is the drift of the function $\bar{v}(x,t) \in C^\infty(0,T; C_0^\infty(\Omega))$ to the points of the mesh $\bar{\omega}_\tau \times \bar{\omega}$. The validity of (53) follows from (9) and the lemma 1. Considering that the function y satisfies equality (21), we rewrite inequality (53) as follows

$$[f_{h\tau}, \hat{y}] + \tau[Ay, y_t] - \sum_{i=1}^n [k_i(x, y\nabla\hat{v}, B_h y), \partial_{r_i}(y - \hat{v})] - \sum_{i=1}^n [k_i(x, y\nabla y, B_h y), \partial_{r_i}\hat{v}] \ge \frac{1}{\tau}[\Phi(\hat{y}) - \Phi(y), 1].$$

Using the extension Π_r^+, we write the last inequality for all $t \in [0,T]$ and integrate the resulting inequality over the segment $[0,t']$, $t' \in [0,T]$. As a result we will have

$$\begin{aligned} J_1(t') \;\; = \;\; & \frac{1}{2^n}\sum_r \int_0^{t'} \{\langle \Pi_r^+ f_{h\tau}, \Pi_r^+ y\rangle - \sum_{i=1}^n \int_\Omega \Pi_r^+ k_i(x,y,\nabla y, B_h y)\Pi_r^+ \partial_{r_i}\hat{v}dx - \\ - \;\; & \sum_{i=1}^n \int_\Omega \Pi_r^+ k_i(x, \nabla\hat{v}, B_h y)\Pi_r^+\partial_{r_i}(y - \hat{v})dx\}dt + \\ + \;\; & \sum_{t=0}^{T-\tau} \tau^2 \mid [Ay, y_t] \mid \ge \frac{1}{2^n}\sum_r \frac{1}{\tau}\int_0^{t'}\int_\Omega \{\Phi(\Pi_r^+\hat{y}) - \Phi(\Pi_r^+ y)\}dx\,dt. \end{aligned} \quad (54)$$

Further, using the [3] methodology, when the condition (48) holds we establish the validity of the limit equality

$$\lim_{\tau,h\longrightarrow 0} \sum_{t=0}^{T-\tau} \tau^2 \big| [Ay, y_t] \big| = 0. \quad (55)$$

Lets notice, that

$$\frac{1}{\tau}\int\limits_{0}^{t'}\int\limits_{\Omega}\{\Phi(\Pi_r^+\hat{y})-\Phi(\Pi_r^+ y)\}dxdt=\frac{1}{\tau}\int\limits_{t'}^{t'+\tau}\int\limits_{\Omega}\Phi(\Pi_r^+ y)dxdt-\int\limits_{\Omega}\Phi(u_0(x))dx.$$

Let further t^* be a mesh point ω_τ, belonging to $(t', t'+\tau]$, $\mu(t')=(t'+\tau-t^*)/\tau$, $\Lambda_\tau-$ linear extension with respect to t. Using the convexity of the function Φ, we have

$$\frac{1}{\tau}\int\limits_{t'}^{t'+\tau}\int\limits_{\Omega}\Phi(\Pi_r^+ y(t))dxdt=\frac{1}{\tau}\left\{\int\limits_{t^*}^{t'+\tau}\int\limits_{\Omega}\Phi(\Pi_r^+ y(t))dxdt+\int\limits_{t'}^{t^*}\int\limits_{\Omega}\Phi(\Pi_r^+ y(t))dxdt\right\}=$$

$$=\mu(t')\int\limits_{\Omega}\Phi(\Pi_r y(t^*))dx+(1-\mu(t'))\int\limits_{\Omega}\Phi(\Pi_r y(t^*-\tau))dx= \tag{56}$$

$$=\int\limits_{\Omega}\left\{\mu(t')\Phi(\Pi_r y(t^*))dx+(1-\mu(t'))\Phi(\Pi_r y(t^*-\tau))\right\}dx\geq$$

$$\geq\int\limits_{\Omega}\Phi(\Pi_r(\mu(t')y(t^*)+(1-\mu(t'))y(t^*-\tau)))dx=\int\limits_{\Omega}\Phi(\Lambda_\tau\Pi_r(y(t')))\,dx.$$

Let us prove further that

$$\Pi_r^+\big(k_i(x,y,\nabla_r\hat{v},B_h y)\big)\ \longrightarrow\ k_i(x,u,\nabla\bar{v},Bu)\quad \text{in}\quad L_{p'}(Q_T). \tag{57}$$

We denote

$$J=\int\limits_{Q_T}\left|\Pi_r^+\big(k_i(x,y,\nabla_r\hat{v},B_h y)\big)-k_i(x,u\nabla\bar{v},Bu)\right|^{p'}dx\,dt\,. \tag{58}$$

Limit relations (45), (47), smoothness of the function v and continuity of $k_i(x,\xi,\eta,v)$ for each of the arguments allow us to assert that the integrand function in (58) tends to 0 as $h,\tau\longrightarrow 0$ almost everywhere in Q_T. In addition, from the estimate (7) it follows that

$$\left|\Pi_r^+\big(k_i(x,y,\nabla_r\hat{v},B_h y)\big)-k_i(x,u,\nabla\bar{v},Bu)\right|^{p'}\leq\left(d_0\sum_{i=1}^{n}\left\{\left|\partial_{r_i}\bar{v}\right|^{p-1}+\left|\frac{\partial\bar{v}}{\partial x_i}\right|^{p-1}\right\}+2\,d_1\right)^{p'}.$$

The right-hand side of the last inequality, due to the smoothness of v is a function integrable over Q_T, therefore, by the Lebesgue theorem on the passage to the limit $J\longrightarrow 0$ for $\tau,\quad h\longrightarrow 0$, it means that (57) holds.

From the inequalities (54)–(56) it follows that

$$\overline{\lim_{\tau,h\to 0}}J_\tau(t')\geq\lim_{\tau,h\longrightarrow 0}\int\limits_{\Omega}\Phi(\Lambda_\tau\Pi_r(y(t'))dx-\int\limits_{\Omega}\Phi(u_0(x))dx. \tag{59}$$

From the relations (42)–(47), (57) it follows that

$$\overline{\lim_{\tau,h\to 0}}J_\tau(t')=\lim_{\tau,h\longrightarrow 0}J_\tau(t')=J(t')\equiv\int\limits_{0}^{t'}\{\langle f,u\rangle-$$

$$-\sum_{i=1}^{n}\int\limits_{\Omega} K_i \frac{\partial v}{\partial x_i} dx - \sum_{i=1}^{n}\int\limits_{\Omega} k_i(x, u\nabla\bar{v}, Bu)\frac{\partial(u-\bar{v})}{\partial x_i} dx\} dt. \tag{60}$$

Considering (51), we will obtain

$$J(t') = \int\limits_{0}^{t'} \{\langle \frac{\partial \varphi(u)}{\partial t}, u\rangle + \sum_{i=1}^{n}\int\limits_{\Omega} (K_i - k_i(x, \nabla\bar{v}, Bu)\frac{\partial(u-\bar{v})}{\partial x_i} dx\} dt. \tag{61}$$

Substituting (60), (61) in the inequality (59) and integrating the result over t' from $T-\lambda$ to T, $\lambda = const > 0$, we will have

$$\int\limits_{T-\lambda}^{T} J(t')dt' \geq \int\limits_{T-\lambda}^{T} \underline{\lim}_{\tau,h\longrightarrow 0} \int\limits_{\Omega} \Phi(\Lambda_\tau \Pi_r(y(t'))dxdt' - \lambda \int\limits_{\Omega} \Phi(u_0(x))dx. \tag{62}$$

The convexity of the function $\Phi(\xi)$ implies the weak lower semicontinuity on $L_\alpha(\Omega)$ of the functional $\int\limits_{\Omega} \Phi(w(x))dx$. Therefore

$$\int\limits_{T-\lambda}^{T} \underline{\lim}_{\tau,h\longrightarrow 0} \int\limits_{\Omega} \Phi(\Lambda_\tau \Pi_r(y(t'))dxdt' \geq \int\limits_{T-\lambda}^{T}\int\limits_{\Omega} \Phi(u(t'))dxdt'. \tag{63}$$

We transform the left-hand side of inequality (62) using the mean value theorem. The application of this theorem is admissible, since the function $J(t')$ is absolutely continuous with respect to t'. Considering (63), we will obtain

$$\lambda J(\bar{t}) = \int\limits_{T-\lambda}^{T}\int\limits_{\Omega} \Phi(u(t'))dxdt' - \lambda\int\limits_{\Omega} \Phi(u_0(x))dx,$$

here $\bar{t} \in [T-\lambda, T]$. We divide both sides of the last inequality by λ and pass to the limit as $\lambda \longrightarrow 0$, as a result we get

$$\int\limits_{0}^{T} \langle \frac{\partial \varphi(u)}{\partial t}, u\rangle dt + \int\limits_{0}^{T}\int\limits_{\Omega} \sum_{i=1}^{n}(K_i - k_i(x, u, \nabla\bar{v}, Bu))\frac{\partial(u-\bar{v})}{\partial x_i} dxdt \geq$$

$$\geq \lim_{\lambda\longrightarrow 0} \frac{1}{\lambda} \int\limits_{T-\lambda}^{T}\int\limits_{\Omega} \Phi(u(t'))dxdt' - \int\limits_{\Omega} \Phi(u_0(x))dx.$$

The last inequality and lemma 3 imply

$$\int\limits_{0}^{T}\int\limits_{0}^{T}\int\limits_{\Omega} \sum_{i=1}^{n}(K_i - k_i(x, u, \nabla\bar{v}, Bu))\frac{\partial(u-\bar{v})}{\partial x_i} dxdt \geq 0. \tag{64}$$

Assuming in the inequality (64) first $\bar{v} = u + \lambda w$, and then $\bar{v} = u - \lambda w$, where $\lambda = const > 0$, w is an arbitrary function from $L_p(0, T; \overset{\circ}{W}{}_p^1(\Omega))$, it is easy to obtain equality (52). The theorem is proved.

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