Computer investigation of the crack problem by the weighted FEM

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Abstract

In present paper we consider crack problem in rectangle. Solution of the problem we define as R_ν-generalized one in the weighted Sobolev set. For calculation of approximate solution we construct the scheme of the weighted finite element method (FEM). We perform comparison of our method with the classic FEM on two model problems. Theoretical estimate of the convergence rate $O(h)$ for constructed method in the norm of the Sobolev weight space is confirmed experimentally.

Keywords

crack problem, angle singularity, R_ν-generalized solution, weighted finite element method

1. Introduction

In different engineering applications it is required to analyze mathematical model for the crack problem. Development of high-precision numerical methods for this problem is actual task.

The crack problem relates of the problems with singularity. Due to the crack, components of the displacements vector belong to the space $W_2^{3/2-\varepsilon}(\Omega)$ when on both crack sides Dirichlet or Neumann boundary conditions are applied, and to the space $W_2^{5/4-\varepsilon}(\Omega)$ when on crack sides different types of boundary conditions are applied. As a result, convergence rate of approximate generalized solution obtained by the classic FEM to the exact one in the norm of the Sobolev space $H^1(\Omega)$ is $O(h^{0.5})$ and $O(h^{0.25})$, respectively.

For elliptic boundary value problems with singularity it was suggested to define solution as R_ν-generalized one in special weighted Sobolev space or set. In what follows properties of R_ν-generalized solutions to the boundary value problems with singularity were studied in detail as well as properties of the weighted spaces and sets [1, 2, 3, 4, 5]. This allowed us to construct new effective numerical methods for such problems [6, 7, 8].

Technique originally suggested to the boundary value problems for elliptic equations with singularity of solution in what follows were extended to the singular boundary value problems in electrodynamics [9], elasticity [10, 11], hydrodynamics [12]. New approaches for solution of boundary value problems in dynamics of fluid-filled pipelines are also under development [13]. Due to introduction of the R_ν-generalized solution, it is succeeded to construct numerical methods with convergence rate $O(h)$ in the norm of the weighted Sobolev space $W_{2,\nu}^1(\Omega)$. In contrast to the special methods developed for such problems, search algorithm for the approximate R_ν-generalized solution is simple and

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natural. It is not required to refine mesh or preliminary separate singularity as multiplicative term.

In the present paper for the crack problem a notion of R_ν-generalized solution in special weighted set is introduced. Such definition of the solution allowed us to construct the weighted FEM with a convergence rate $O(h)$ in the norm of the weighted Sobolev space. A comparison of the constructed method with the classic FEM is carried out. Theoretical estimation of the convergence rate for the constructed method derived in [14] is confirmed in numerical experiment for two model problems. It is twice as much than for the classic FEM. At the same time, in the most of mesh nodes the absolute error for the approximate R_ν-generilized solution in a several decimal orders less than for the approximate generalized solution.

2. Weighted spaces and sets. R_ν-generalized solution

Let $\Omega = (-0.7, 0.3)\times[-1, 1]\setminus[0, 0.3]\times\{0\}$ be a two-dimensional domain with a crack $\partial\Omega_c = [0, 0.3]\times\{0\}$, $\partial\Omega_c^+$ and $\partial\Omega_c^-$ are the crack sides, point $(0, 0)$ is a crack tip. Denote $\partial\Omega$ the boundary of Ω, $\partial\Omega_c \subset \partial\Omega$.

Assume that the domain Ω is a homogeneous isotropic body and strains are small. Consider the following boundary value problem of elasticity stated in displacements (crack problem):

$$-\left(2\,\mathbf{div}(\mu\varepsilon(\mathbf{u})) + \mathbf{grad}(\lambda\,\mathrm{div}\,\mathbf{u})\right) = \mathbf{f}, \quad x \in \Omega, \tag{1}$$

$$u_i = q_i, \;\; i = 1, 2, \quad x \in \partial\Omega. \tag{2}$$

Here $\mathbf{u} = (u_1, u_2)$ is a displacement field, $\varepsilon(\mathbf{u})$ is a strain tensor, $\mathbf{f} = (f_1, f_2)$ is a distributed body force, q_i, $i = 1, 2$ are components of a surface force vector, λ and μ are Lamé parameters.

Denote by Ω' closure of the δ-neighborhood of the point $(0, 0)$ in the domain Ω: $\Omega' = \{x \in \overline{\Omega} : \sqrt{(x_1^2 + x_2^2)} \le \delta\}$. In Ω' we introduce a weight function $\rho(x)$ as a distance to the point $(0, 0)$ and extend it to the rest of $\overline{\Omega}$ with constant δ. Using $\rho(x)$, we introduce weighted space $L_{2,\nu}(\Omega)$ consisting of Lebesque measurable functions u with finite norm $\|u\|_{L_{2,\alpha}(\Omega)} = \sqrt{\int_\Omega \rho^{2\alpha} u^2 dx}$, $\alpha \in R$, $\alpha \ge 0$. Through $W^1_{2,\alpha}(\Omega, \delta)$ we denote the set of functions u that meet the following conditions:

(a) $u \le c_1 \delta^\alpha \rho^{-\alpha}$, $x \in \Omega'$,

(b) $\left|\frac{\partial u}{\partial x_i}\right| \le c_1 \delta^{\alpha+1} \rho^{-\alpha-1}$, $x \in \Omega'$, $i = 1, 2$, $c_1 = \mathrm{const}$,

(c) $\|u\|_{L_{2,\alpha}(\Omega\setminus\Omega')} \ge c_2$, $c_2 = \mathrm{const}$,

with finite norm $\|u\|_{W^1_{2,\alpha}(\Omega)} = \left(\|u\|^2_{L_{2,\alpha}(\Omega)} + \left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_{2,\alpha}(\Omega)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_{2,\alpha}(\Omega)}\right)^{1/2}$.

For the vector function $\mathbf{u} = (u_1, u_2)$ corresponding norm is calculated by formula

$$\|\mathbf{u}\|_{W^1_{2,\alpha}(\Omega)} = \left(\|u_1\|^2_{W^1_{2,\alpha}(\Omega)} + \|u_2\|^2_{W^1_{2,\alpha}(\Omega)}\right)^{1/2}.$$

The set consisting of traces of functions $u \in W^1_{2,\alpha}(\Omega, \delta)$ on $\partial\Omega$ we denote by $W^{1/2}_{2,\alpha}(\partial\Omega, \delta)$:

$$W^{1/2}_{2,\alpha}(\partial\Omega, \delta) = \{g : \; g = G|_{\partial\Omega}, G \in W^1_{2,\alpha}(\Omega, \delta)\}.$$

Norm in this set is defined by formula

$$\|g\|_{W^{1/2}_{2,\alpha}(\partial\Omega,\delta)} = \inf_{G|_{\partial\Omega}=g} \|G\|_{W^1_{2,\alpha}(\Omega)}.$$

The subset of set $W^1_{2,\alpha}(\Omega, \delta)$ that contains functions with zero trace on $\partial\Omega$ we denote by $\mathring{W}^1_{2,\alpha}(\Omega, \delta)$.

The set of functions $u \in L_{2,\alpha}(\Omega)$ satisfying conditions (a) and (c) we denote by $L_{2,\alpha}(\Omega, \delta)$.

We assume that for some real number $\beta > 0$ the following inequalities for the right hand sides of equations (1) and boundary conditions (2) are satisfied:

$$f_i \in L_{2,\beta}(\Omega, \delta), \;\; q_i \in W^{1/2}_{2,\beta}(\partial\Omega, \delta), \;\; i = 1, 2. \tag{3}$$

We introduce bilinear and linear forms, respectively:

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega 2\mu\varepsilon(\mathbf{u}) : \; \varepsilon(\rho^{2\nu}\mathbf{v}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div}(\rho^{2\nu}\mathbf{v})dx, \quad l(\mathbf{v}) = \int_\Omega \rho^{2\nu}\mathbf{f} \cdot \mathbf{v}dx.$$

Vector-function $\mathbf{u}_\nu = (u_{\nu,1}, u_{\nu,2})$ with components $u_{\nu,i} \in W^1_{2,\nu}(\Omega, \delta)$, $i = 1, 2$, is called **$\mathbf{R}_\nu$-generalized solution** of the problem (1), (2) if on $\partial\Omega$ boundary conditions (2) are satisfied almost everywhere, and for any vector-function $\mathbf{v} = (v_1, v_2)$, $v_i \in \mathring{W}^1_{2,\nu}(\Omega, \delta)$, $i = 1, 2$, the integral identity

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v})$$

holds for any fixed value of $\nu \geq \beta$.

3. The scheme of the weighted FEM

We construct the scheme of the weighted FEM for calculation of approximate R_ν-generalized solution for the problem (1), (2).

We perform quasiuniform triangulation T^h of the domain Ω coordinated with the crack $\partial\Omega_c$. For this, we decompose Ω into the set of rectangles by horizontal and vertical straight lines $x = x_i$, $y = y_j$, $x_i \in [-0.7, 0.3]$, $y_j \in [-1, 1]$. Each rectangle we divide into two triangles by diagonal. Obtained triangles K we call finite elements and their vertices are nodes P_i, $i = 1, \dots, N$. The set of internal nodes we designate $\{P_i\}_{i=1}^n$, and $\{P_i\}_{i=n}^N$ is the set of boundary nodes. The longest side of all triangles we designate h and call mesh parameter.

For each node P_i we introduce weighted basis function $\psi_i(x) = \rho^{\nu^*}(x)\varphi_i(x)$, $i = 1, \dots, N$, $\nu^* \in R$, where $\varphi_i(x)$ is a function linear on each finite element K and $\varphi_i(x)$ is equal 1 in P_i and equal 0 in all other nodes. Linear span of all built basis function we designate V^h. Linear span of basis function associated with internal nodes we designate $\mathring{V}^h$.

Vector-function $\mathbf{u}^h_\nu = (u^h_{\nu,1}, u^h_{\nu,2})$ with components $u^h_{\nu,i} \in V^h$, $i = 1, 2$, is called **approximate $\mathbf{R}_\nu$-generalized solution** of the problem (1), (2) if its components $u^h_{\nu,i}$, $i = 1, 2$ satisfy boundary conditions (2) on $\partial\Omega$, and for any vector-function $\mathbf{v}^h = (v^h_1, v^h_2)$, $v^h_i \in \mathring{V}^h$, $i = 1, 2$, integral identity

$$a(\mathbf{u}^h_\nu, \mathbf{v}^h) = l(\mathbf{v}^h)$$

holds.

Components of approximate R_ν-generalized solution we write in the form

$$u^h_{\nu,1} = \sum_{i=1}^{n} d_{2i-1}\psi_i, \quad u^h_{\nu,2} = \sum_{i=1}^{n} d_{2i}\psi_i.$$

Unknown coefficients d_i, $i = 1, \dots, 2n$ can be found from the system of linear algebraic equations

$$\begin{cases} a(\mathbf{u}^h_\nu, (\psi_i, 0)) = l(\psi_i, 0), \\ a(\mathbf{u}^h_\nu, (0, \psi_i)) = l(0, \psi_i), i = 1, \dots, n. \end{cases}$$

Remark. The difference of the weighted FEM is the weight function $\rho(x)$ raise to some power ν^* in the finite element basis. This allowed us to approximate behavior of the solution near the singularity point better. Parameter ν^* as well as radius δ of the δ-neighborhood in definition of the weight function $\rho(x)$ and its power ν in definition of the R_ν-generalized solution are governing parameters of the constructed weighted FEM. Varying these parameters, we are able to affect accuracy of the approximate R_ν-generalized solution. When we choose parameters close to the optimal ones, we get the best accuracy and maximum convergence rate of the constructed FEM that corresponds to the theoretical rate $O(h)$.

4. Computer investigation of the model problems

In present section we adduce computer investigation of two model problems using weighted FEM constructed in section 3. The results were obtained using the equipment of Shared Resource Center "Far Eastern Computing Resource" IACP FEB RAS (https://cc.dvo.ru) and of Shared Services Center “Data Center of FEB RAS” (Khabarovsk). Computation of approximate R_ν-generalized solution were realized by the program "Proba-IV", automatic startup of calculation series and sequent analysis of results were carried out by the software package [15].

The common algorithm of investigation is the following:

1. Selection of the exact solution $\mathbf{u}$, substitution to the equation (1) and computation of the right hands of equation and boundary conditions (2).
2. Calculation of approximate R_ν-generalized solution by the weighted FEM on the series of meshes with decreasing mesh parameter h and with different values of governing parameters of the method. To do this, on each mesh we realized the following:
   a) Definition of ranges for governing parameters δ, ν, ν^*, their increasing steps and creation of sets of their fixed values.
   b) Computation of the approximate R_ν-generalized solution corresponding to the each created set of values of governing parameters δ, ν, ν^*.
   c) Collection of the derived results.
   d) Analysis of results and detection of the optimal parameters set that minimize the computational error in the norm of the weighted space $W^1_{2,\nu}(\Omega)$.
3. Experimental evaluation of the convergence rate of approximate R_ν-generalized solution by the weighted FEM calculated with optimal values of governing parameters δ, ν, ν^* found in listbox 2d), comparison with the convergence rate of the approximate generalized solution derived by the classic FEM.
4. Comparison of the absolute error in the mesh nodes of approximate R_ν-generalized and generalized solutions.

4.1. Model problems

Problem 1. For the problem 1 we choose the vector $\mathbf{u}$ with only singular components:

$$u_1 = \frac{K_I}{\mu\sqrt{2\pi}} \cos(x_1)\cos^2(x_2)(x_1^2 + x_2^2)^{0.3051},$$
$$u_2 = \frac{K_I}{\mu\sqrt{2\pi}} \cos^2(x_1)\cos(x_2)(x_1^2 + x_2^2)^{0.3051}.$$

Table 1
Dependence of relative error ϑ_ν for approximate R_ν-generalized solution derived with indicated optimal parameters δ, ν, ν^*, and of relative error ϑ for approximate generalized solution on the mesh parameter h, model problem 1.

h	0.062	k	0.031	k	0.0155	k	0.0077	k	0.0038	k	0.0019
δ	0.062		0.062		0.046		0.023		0.0116		0.0058
ν	2.0		1.3		1.2		1.2		1.6		1.6
ν^*	0.1		0.1		0.1		0.1		0.2		0.2
ϑ_ν	$1.208 \cdot 10^{-1}$	2.12	$5.693 \cdot 10^{-2}$	1.97	$2.891 \cdot 10^{-2}$	1.99	$1.452 \cdot 10^{-2}$	2.00	$7.247 \cdot 10^{-3}$	2.00	$3.619 \cdot 10^{-3}$
ϑ	$2.827 \cdot 10^{-1}$	1.41	$2.005 \cdot 10^{-1}$	1.41	$1.420 \cdot 10^{-1}$	1.41	$1.005 \cdot 10^{-1}$	1.41	$7.106 \cdot 10^{-2}$	1.41	$5.026 \cdot 10^{-2}$

Problem 2. For the problem 2 we choose the vector $\mathbf{u}$ with both singular and regular components:

$$u_1 = \frac{K_I}{\mu\sqrt{2\pi}} \left(\cos(x_1)\cos^2(x_2)(x_1^2 + x_2^2)^{0.3051} + (x_1^2 + x_2^2)\right),$$
$$u_2 = \frac{K_I}{\mu\sqrt{2\pi}} \left(\cos^2(x_1)\cos(x_2)(x_1^2 + x_2^2)^{0.3051} + (x_1^2 + x_2^2)\right).$$

Lamé parameters for both problems are $\lambda = 578.923$, $\mu = 384.615$Pa, stress intensity factor $K_I = 1.611$.

4.2. Investigation of the convergence rate

In the present subsection for two model problems we adduce results concerning convergence rate of the approximate R_ν-generalized solution calculated by the weighted FEM with optimal parameters. For derived approximate R_ν-generalized solutions $\mathbf{u}_\nu^h$ the error was calculated in the relative weighted norm of the weighted space $W^1_{2,\nu}(\Omega)$ when parameter $\nu = \overline{\nu} = 2.2$, $\delta = \overline{\delta} = 0.062$ by formula

$$\vartheta_\nu = \frac{\|\mathbf{u} - \mathbf{u}_\nu^h\|_{W^1_{2,\overline{\nu}}(\Omega)}}{\|\mathbf{u}\|_{W^1_{2,\overline{\nu}}(\Omega)}}.$$

For approximate generalized solutions $\mathbf{u}^h$ the error was calculated in the norm of the space $W^1_2(\Omega)$ by formula

$$\vartheta = \frac{\|\mathbf{u} - \mathbf{u}^h\|_{W^1_2(\Omega)}}{\|\mathbf{u}_\nu\|_{W^1_2(\Omega)}}.$$

For the model problem 1 in Table 1 for meshes with different parameter h we adduce values of ϑ_ν, corresponding optimal parameters δ, ν, ν^*, values of ϑ and their ratios k for adjacent meshes. Adduced results confirm theoretical estimations of convergence rate $O(h^{0.5})$ for approximate generalized solution, and $O(h)$ for approximate R_ν-generalized one.

On the Figure 1 we present graphs of ϑ_ν and ϑ in the log scales.

Results on the convergence rate for model problem 2 are presented in Table 2 and Figure 2.

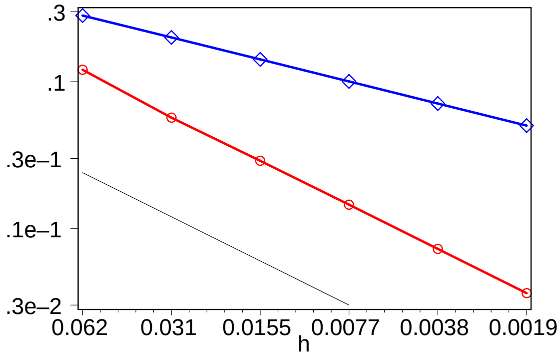

Figure 1: Graphs of ϑ_ν (red circled line) and ϑ (blue rhombic line) in log scales, model problem 1. Solid black line represents convergence rate $O(h)$.

Table 2
Dependence of relative error ϑ_ν for approximate R_ν-generalized solution derived with indicated optimal parameters δ, ν, ν^*, and of relative error ϑ for approximate generalized solution on the mesh parameter h, model problem 2.

h	0.062	k	0.031	k	0.0155	k	0.0077	k	0.0038	k	0.0019
δ	0.062		0.062		0.046		0.023		0.0116		0.0058
ν	2.0		1.3		1.2		1.2		1.6		1.6
ν^*	0.1		0.1		0.1		0.1		0.2		0.2
ϑ_ν	$6.191 \cdot 10^{-2}$	2.09	$2.956 \cdot 10^{-2}$	1.98	$1.494 \cdot 10^{-2}$	1.99	$7.495 \cdot 10^{-3}$	2.00	$3.742 \cdot 10^{-3}$	2.00	$1.869 \cdot 10^{-3}$
ϑ	$1.376 \cdot 10^{-1}$	1.43	$9.627 \cdot 10^{-2}$	1.42	$6.768 \cdot 10^{-2}$	1.42	$4.772 \cdot 10^{-2}$	1.42	$3.369 \cdot 10^{-2}$	1.41	$2.381 \cdot 10^{-2}$

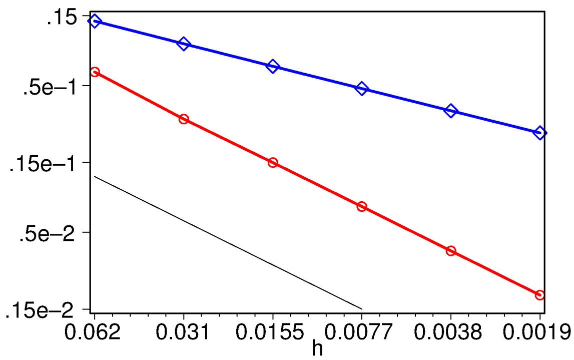

Figure 2: Graphs of ϑ_ν (red circled line) and ϑ (blue rhombic line) in log scales, model problem 2. Solid black line represents convergence rate $O(h)$.

Table 3
Values of n_1^ν, n_1 (in percents of total number of nodes) on meshes with different parameter h, model problem 1.

h	0.062	0.031	0.0155	0.0077	0.0038	0.0019
n_1^ν, %	9.786	45.179	66.753	77.184	90.734	97.123
n_1, %	1.584	2.360	7.453	15.707	30.164	50.187

Table 4
Values of n_2^ν, n_2 (in percents of total number of nodes) on meshes with different parameter h, model problem 1.

h	0.062	0.031	0.0155	0.0077	0.0038	0.0019
n_2^ν, %	4.567	33.648	44.962	72.075	94.659	99.281
n_2^ν, %	0.186	1.349	4.533	9.101	15.574	24.859

Table 5
Values of n_1^ν, n_1 (in percents of total number of nodes) on meshes with different parameter h, model problem 2.

h	0.062	0.031	0.0155	0.0077	0.0038	0.0019
n_1^ν, %	9.786	45.179	66.753	77.184	90.734	97.123
n_1, %	1.584	2.360	7.453	15.707	30.164	50.187

Table 6
Values of n_2^ν, n_2 (in percents of total number of nodes) on meshes with different parameter h, model problem 2.

h	0.062	0.031	0.0155	0.0077	0.0038	0.0019
n_2^ν, %	4.567	33.648	44.962	72.075	94.659	99.281
n_2^ν, %	0.186	1.349	4.533	9.101	15.574	24.859

4.3. Investigation of absolute error in mesh nodes

In nodes P_i, $i = 1, \ldots, N$ on meshes with different parameter h for components of approximate R_ν-generalized solution obtained with optimal values of parameters δ, ν, ν^* and for components of approximate generalized solutions we calculated the absolute differences between them and components of exact solution: $\delta_{ij}^\nu = \left|u_j(P_i) - u_{\nu,j}^h(P_i)\right|$, $\delta_{ij} = \left|u_j(P_i) - u_j^h(P_i)\right|$, $j = 1, 2$. We also counted the numbers n_j^ν, n_j of nodes where the absolute errors δ_{ij}^ν and δ_{ij}, respectively, are less than the limit value $\overline{\Delta} = 5 \cdot 10^{-8}$.

For model problem 1, in Table 3 we adduce values of n_1^ν, n_1 on meshes with different parameter h. Results for second component of approximate R_ν-generalized and generalized solution are presented in Table 4.

On Figure 3 and Figure 4 for model problem 1 we depict distribution in the domain Ω of δ_{i1}^ν, δ_{i1} and δ_{i2}^ν, δ_{i2}, respectively.

Values of n_1^ν, n_1 and n_2^ν, n_2 for model problem 2 are presented in Table 5 and Table 6 respectively.

Distribution in the domain Ω of δ_{i1}^ν, δ_{i1} and δ_{i2}^ν, δ_{i2} for model problem 2 is fully similar to the model problem 1.

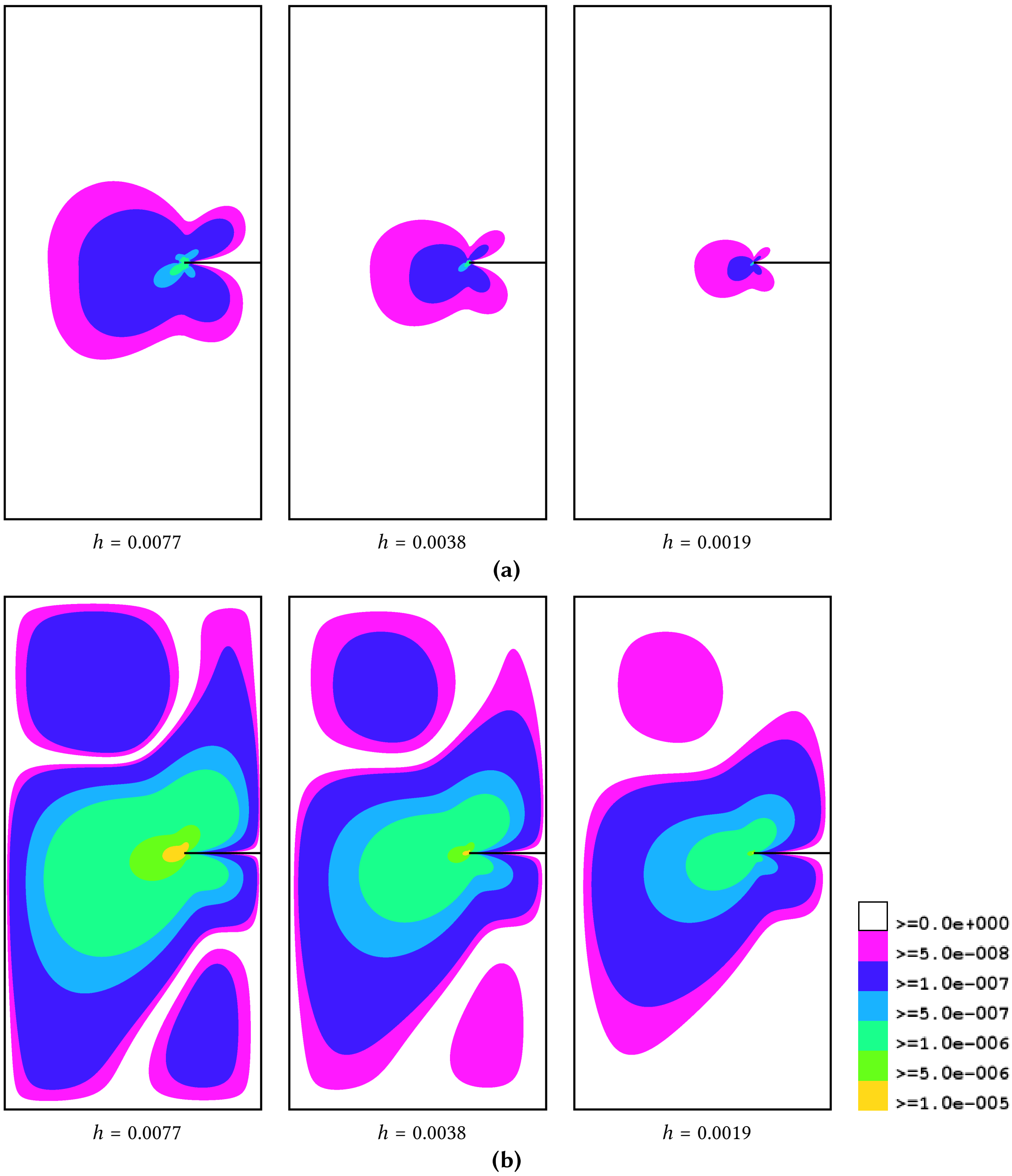

Figure 3: Distribution of the absolute errors δ_{i1}^{v} (a) and δ_{i1} (b) for model problem 1.

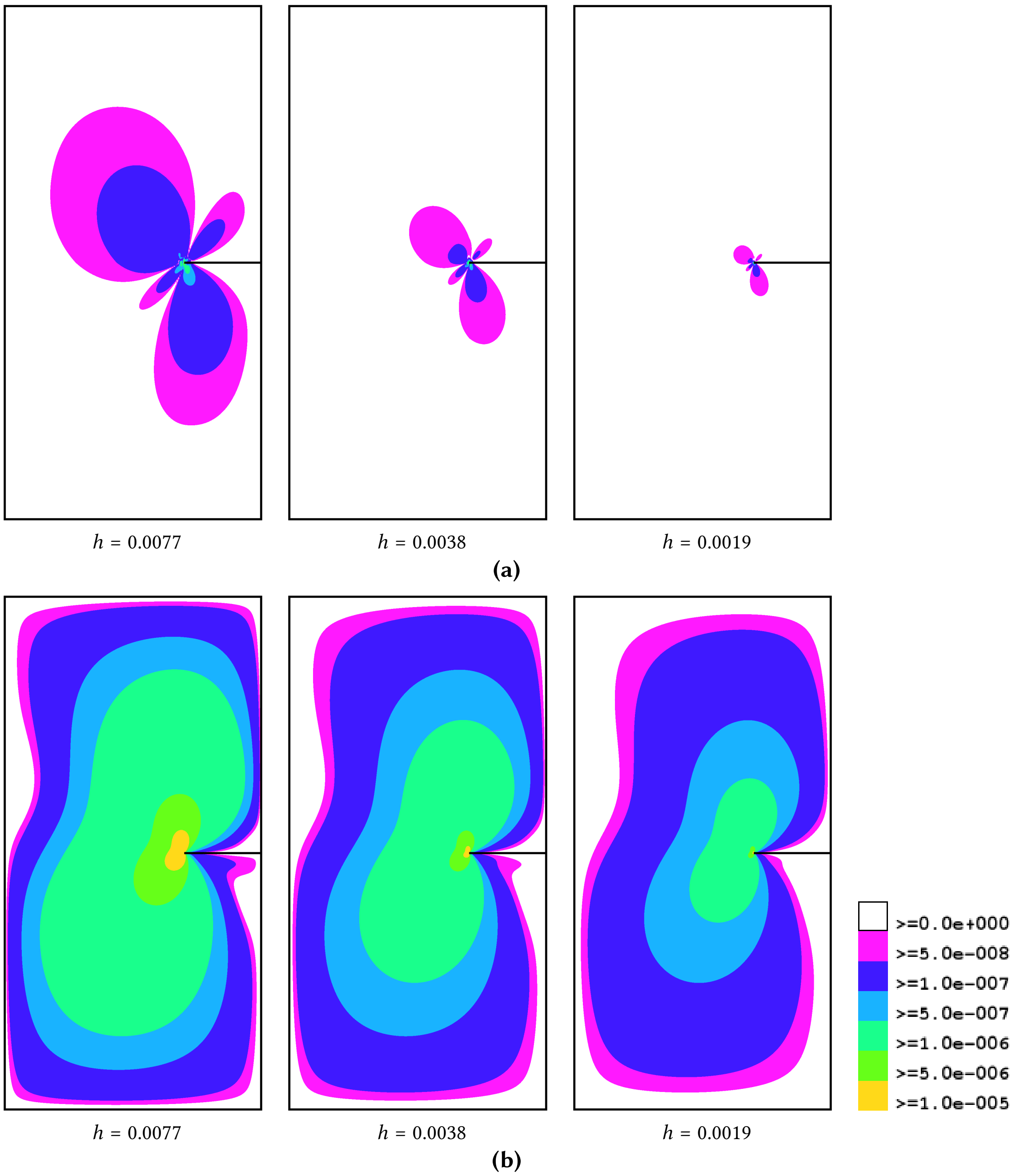

Figure 4: Distribution of the absolute errors δ_{i2}^{v} (a) and δ_{i2} (b) for model problem 1.

5. Conclusion

Computer investigation of the crack problem realized by the weighted FEM allows us to draw following conclusions:

- Theoretical convergence rate $O(h)$ of the approximate R_ν-generalized solution by the weighted FEM with optimal governing parameters to the exact one in the norm of the weighted space $W^1_{2,\nu}(\Omega)$ were experimentally confirmed. This is twice as much than convergence rate of the approximate generalized solution.

- In most mesh nodes, absolute error of approximate R_ν-generalized solution by the weighted FEM in several decimal orders less than absolute error of approximate generalized solution.

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