On the convergence of an approximate method for solving the problem filtration consolidation with a limiting gradient

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Abstract

A one-dimensional initial-boundary value problem modelling the process of joint motion of a viscoelastic porous medium and a liquid saturating the medium is considered. In the filtration theory, this process is called filtration consolidation. A finite element in spatial variable and time-implicit difference scheme is constructed. Its solvability is established, the convergence piecewise-constant filling of an approximate solution in the variable t to generalized solution of problem is proved.

Keywords

filtration, filtration consolidation, difference scheme, finite element method

1. Problem statement

An initial-boundary value problem is considered, which is described by the following system of nonlinear partial differential equations for the unknown functions $u(x,t)$, $p(x,t)$:

$$-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x \partial t}\right)+\frac{\partial p}{\partial x}=f(x,t), \quad 0<x<L,\ 0<t<T, \tag{1}$$

$$\frac{\partial^2 u}{\partial x \partial t}-\frac{\partial}{\partial x}\left(g\left(\left|\frac{\partial p}{\partial x}\right|\right)\frac{\partial p}{\partial x}\right)=0, \quad 0<x<L,\ 0<t<T. \tag{2}$$

We assume that for $t \in (0, T]$ the following boundary conditions are satisfied

$$u(0,t)=0, \quad \frac{\partial u}{\partial x}(L,t)+\frac{\partial^2 u}{\partial x \partial t}(L,t)=0, \tag{3}$$

$$p(0,t)=p(L,t)=0. \tag{4}$$

The initial conditions are given as

$$u(x,0)=u_0(x), \quad p(x,0)=p_0(x), \quad 0 \le x \le L. \tag{5}$$

The problem (1)–(5) is of an applied nature: relations (1)–(5) can be used to describe an one-dimensional process of filtration consolidation with a limiting gradient (see, eg, [1]). In this case, p is liquid pressure in the pores, u is motion of the skeleton particles, $f(x,t)$, $g(\xi)$ are given functions.

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The foundations of the theory of filtration consolidation were laid in such works as [2, 3, 4, 5]. In these works mathematical models of filtration consolidation were built, and studies of the models from the standpoint of continuum mechanics were carried out. A rigorous mathematical analysis of problems of filtration consolidation was carried out in [6, 7], where the solvability of these problems in the class of generalized functions is established. The works [8, 9, 10] are devoted to experimental study using numerical methods.

This paper considers the problem of filtration consolidation with limiting gradient in the case when the function g is defining the law filtration as follows:

$$g(|\xi|) = \begin{cases} 0, & |\xi| \leq \xi_0, \\ 1, & |\xi| > \xi_0. \end{cases}$$

In what follows, we assume that the functions $g(\xi)$, $f(x,t)$ satisfy the following conditions:
A_1. $g(\xi)$, $\xi \geq 0$ is an absolutely continuous in ξ, nonnegative, nondecreasing function and there exist $\xi_0 \geq 0$, η, $\mu > 0$, such that at $\xi \geq \xi_0$ the following inequality holds

$$\eta(\xi - \xi_0) \leq g(|\xi|)\xi \leq \mu(\xi - \xi_0). \tag{6}$$

A_2. The function $f(x,t)$ is continuous at $(x,t) \in Q_T$, where $Q_T = (0,L) \times (0,T]$.

Conditions (6) imposed on the function g mean that the filtration rate will be zero for small values of the gradient modulus.

2. Defining a generalized solution

Let $\overset{\circ}{V}$ be the closure of smooth functions equal to zero at $x = 0$ in the norm of the space $W_2^{(1)}(0,L)$, and let $\overset{\circ}{V}_1$ be the closure of smooth functions equal to zero on the boundary of the interval $[0,L]$, in the norm of the same space.

Definition. By a generalized solution to problem (1)–(5), we imply functions (u,p), for which the following conditions hold:

$$u \in W_2^{(1)}(0,T;\overset{\circ}{V}), \quad p \in L_2(0,T;\overset{\circ}{V}_1),$$

$$u(x,0) = u_0(x), \quad p(x,0) = p_0(x) \quad \text{almost everywhere on} \quad x \in (0,L),$$

and for any functions $v \in W_2^{(1)}(0,T;\overset{\circ}{V})$, $z \in L_2(0,T;\overset{\circ}{V}_1)$ the following equality is true:

$$\int_0^T \int_0^L \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} \right) \frac{\partial^2 v}{\partial x \partial t} - p \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} z + \right.$$
$$\left. + g\left(\left| \frac{\partial p}{\partial x} \right| \right) \frac{\partial p}{\partial x} \frac{\partial z}{\partial x} \right\} dxdt = \int_0^T \int_0^L f(x,t) \frac{\partial v}{\partial t} dxdt. \tag{7}$$

3. The discrete problem

The problem (1)–(5) will be solved by the semi-discretization method in combination with the finite element method. Let us construct an arbitrary unequally-spaced grid on the interval [0, L]

$$\bar{\omega}_h = \{x_0 = 0 < x_1 < \ldots < x_n = L\}.$$

Let V^n and V_1^n be finite-dimensional function spaces, continuous on the interval $[0, L]$, satisfying the conditions (3) and (4), respectively, and are polynomials of the first degree on each grid cell $\delta_i = [x_{i-1}, x_i]$, $i = 1, 2, ..., n$. Let also

$$\bar{\omega}_\tau = \{t = k\tau,\ 0 \le k \le M,\ M\tau = T\},$$

$\omega_\tau = \bar{\omega}_\tau \setminus \{0\}$.

Definition. By the approximate solution to the problem (1)–(5) constructed by the method of semi-discretization in combination with the finite element method, we imply the functions $(\hat{u}^n(t), \hat{p}^n(t))$ for which the following conditions hold:

$$\hat{u}^n(t) \in V^n, \quad \hat{p}^n(t) \in V_1^n \qquad \forall t \in \omega_\tau,$$

$$u^n(x, 0) = u_0(x), \quad p^n(x, 0) = p_0(x) \quad \text{almost everywhere on} \quad x \in (0, L),$$

and for any functions $v^n \in V^n$, $z^n \in V_1^n$ the following equality is true

$$\int_0^L \left\{ \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial v_t^n}{\partial x} - \hat{p}^n \frac{\partial v_t^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \hat{z}^n + g\left(\left| \frac{\partial \hat{p}^n}{\partial x} \right| \right) \frac{\partial \hat{p}^n}{\partial x} \frac{\partial \hat{z}^n}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t)\, v_t^n dx. \quad (8)$$

Here $\hat{v} = v(t + \tau)$, $v_t = \dfrac{\hat{v} - v}{\tau}$.

Theorem 1. Approximate solution of the problem (1)–(5) exists.

Proof. Obviously, it suffices to establish the existence of $\hat{p}^n$, $\hat{u}^n$ satisfying (8), under the assumption that p^n, u^n are known.

Since the choice of the functions v^n, z^n is arbitrary, the equality (8) is equivalent to the following system

$$\int_0^L \left\{ \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial v_t^n}{\partial x} - \hat{p}^n \frac{\partial v_t^n}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t)\, v_t^n dx, \quad (9)$$

$$\int_0^L \left\{ \frac{\partial u_t^n}{\partial x} \hat{z}^n + g\left(\left| \frac{\partial \hat{p}^n}{\partial x} \right| \right) \frac{\partial \hat{p}^n}{\partial x} \frac{\partial \hat{z}^n}{\partial x} \right\} dx = 0. \quad (10)$$

We will look for approximate solutions in the form

$$\hat{u}^n = \sum_{k=0}^n \zeta_k^{(1)} \varphi_k, \qquad \hat{p}^n = \sum_{k=0}^n \zeta_k^{(2)} \psi_k,$$

where φ_k, ψ_k are linear on each element, continuous on the interval $[0, L]$ functions satisfying the conditions

$$\varphi_k(x_j) = \begin{cases} 1, & k = j, \\ 0, & k \ne j,\ k = 0, 1, \dots, n, \end{cases} \qquad \psi_k(x_j) = \begin{cases} 1, & k = j, \\ 0, & k \ne j, \quad k = 0, 1, \dots, n. \end{cases}$$

The unknown coefficients $\zeta_k^{(i)}$, $k = 1, 2, \dots, n$ $i = 1, 2$ are determined by the following system of equations:

$$\int_0^L \left\{ \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial (\varphi_k)_t}{\partial x} - \hat{p}^n \frac{\partial (\varphi_k)_t}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t)\, (\varphi_k)_t dx, \quad (11)$$

$$\int_0^L \left\{ \frac{\partial u_t^n}{\partial x}\, \hat{\psi}_k + g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right) \frac{\partial \hat{p}^n}{\partial x}\, \frac{\partial \hat{\psi}_k}{\partial x} \right\} dx = 0. \tag{12}$$

Let $\mathbf{H} : R^{2n} \longrightarrow R^{2n}$ be a nonlinear operator such that the equation

$$\mathbf{H}(\zeta) = 0$$

is equivalent to system (11)–(12). Let us make sure that R^{2n} contains a sphere centered at zero of finite radius, on which

$$(\mathbf{H}(\zeta), \zeta)_{R^{2n}} \geq 0. \tag{13}$$

We have

$$(\mathbf{H}(\zeta), \zeta)_{R^{2n}} = \int_0^L \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial u_t^n}{\partial x} dx + \\ + \int_0^L g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right) \left(\frac{\partial \hat{p}^n}{\partial x}\right)^2 dx - \int_0^L \hat{f}(x,t)\, u_t^n dx. \tag{14}$$

The first term on the right-hand side of equality (14) can be transformed to the form:

$$\int_0^L \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial u_t^n}{\partial x}\, dx \;=\; \left(\frac{1}{\tau} + \frac{1}{\tau^2}\right) \parallel \hat{u}^n \parallel_1^2 - \left(\frac{1}{\tau} + \frac{2}{\tau^2}\right) \int_0^L \frac{\partial \hat{u}^n}{\partial x}\, \frac{\partial u^n}{\partial x}\, dx + \frac{1}{\tau^2} \parallel u^n \parallel_1^2 . \tag{15}$$

Here $\parallel v \parallel_1^2 = \int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx.$

Using the Cauchy-Bunyakovsky inequality

$$(x, y) \;\leq\; \delta \|x\|^2 \;+\; \frac{1}{4\delta}\|y\|^2, \tag{16}$$

from (15) it is easy to obtain the following estimate

$$\int_0^L \left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x} \right) \frac{\partial u_t^n}{\partial x}\, dx \geq \left(\frac{1}{\tau} + \frac{1}{\tau^2} - \delta\right) \parallel \hat{u}^n \parallel_1^2 - \frac{1}{4\delta}\left(\frac{1}{\tau} + \frac{2}{\tau^2}\right)^2 \parallel u^n \parallel_1^2 . \tag{17}$$

Using (16) and inequality (6), for the second term in equality (14) we have

$$\int_0^L g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right) \left(\frac{\partial \hat{p}^n}{\partial x}\right)^2 dx \geq (\eta - \delta) \parallel \hat{p}^n \parallel_1^2 - \frac{\eta^2 \xi_0^2 L^2}{4\delta}.$$

To estimate the last term of (14), we use the boundedness of the function $f(\xi)$, inequality (16), and the Friedrichs inequality. As a result, we obtain

$$\left| \int_0^L \hat{f}(x,t) \cdot u_t^n dx \right| \leq \delta \parallel \hat{u}^n \parallel_1^2 + \frac{C^2 C_F^2 L^2}{4\delta\tau^2} + \frac{C C_F}{\tau} \parallel u^n \parallel_1^2,$$

here C_F is a constant of the Friedrichs inequality, C is a constant such that

$$|f(\xi,\zeta)| \le C \qquad \forall \xi \in [0,L], \quad \forall \zeta \in [0,T].$$

Substituting the estimates obtained in (14), we have

$$(\mathbf{H}(\zeta),\zeta)_{R^{2n}} \ge \overline{K}(\delta)\left(\| \hat{u}^n \|_1^2 + \| \hat{p}^n \|_1^2\right) - \overline{R}(\delta)\ , \tag{18}$$

where

$$\overline{K}(\delta) = \min\left\{\left(\frac{1}{\tau} + \frac{1}{\tau^2} - 2\delta\right), \eta - \delta\right\},$$

$$\overline{R}(\delta) = \left(\frac{CC_F}{\tau} + \frac{1}{4\delta}\left(\frac{1}{\tau} + \frac{2}{\tau^2}\right)^2\right) \| u^n \|_1^2 + \frac{C^2C_F^2L^2}{4\delta\tau^2} + \frac{\eta^2\xi_0^2L^2}{4\delta}.$$

Let δ^* be a constant such that for all $0 < \delta \le \delta^*$ the following inequality holds

$$\overline{K}(\delta) \ \ge \ \beta = const > 0\ ,$$

and $S \subset R^{2n}$ be a sphere centered at zero at which the right-hand side of inequality (18) is non-negative. Then, by the topological lemma ([11], p. 66), there is at least one solution to the system inside this sphere. The proof of Theorem 1 is complete.

Lemma 1. For the approximate solution (8), the following a priori estimates are valid

$$\max_{t'}\|u^n(t')\|_1^2 \le C, \qquad \sum_{t=0}^{t'} \tau\|p^n(t)\|_1^2 \le C, \tag{19}$$

$$\sum_{t=0}^{t'-\tau} \tau\,\|(u^n)_t\|_1^2 \le C, \qquad \sum_{t=0}^{t'} \tau\left\|\frac{\partial u_t^n}{\partial x}\right\|_{L_2(0,L)}^2 \le C, \tag{20}$$

$$\sum_{t=0}^{t'-\tau} \tau\left\|g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right)\frac{\partial \hat{p}^n}{\partial x}\right\|_{L_2(0,L)}^2 \le C. \tag{21}$$

Proof. Let us assume in (8) $v^n = u^n$, $z^n = p^n$ and obtain

$$\int_0^L\left\{\left(\frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u_t^n}{\partial x}\right)\frac{\partial u_t^n}{\partial x} + g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right)\left(\frac{\partial \hat{p}^n}{\partial x}\right)^2\right\}dx = \int_0^L \hat{f}(x,t)\,u_t^n\,dx. \tag{22}$$

Note that

$$\frac{\partial \hat{u}^n}{\partial x}\frac{\partial u_t^n}{\partial x} = \frac{\partial \hat{u}^n}{\partial x}\frac{\partial}{\partial x}\left(\frac{\hat{u}^n - u^n}{\tau}\right) = \frac{\partial \hat{u}^n}{\partial x}\cdot\frac{1}{\tau}\left(\frac{\partial \hat{u}^n}{\partial x} - \frac{\partial u^n}{\partial x}\right) =$$
$$= \frac{1}{2}\left(\frac{\partial \hat{u}^n}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial u^n}{\partial x}\right)^2 + \frac{\tau^2}{2}\left(\frac{\partial u_t^n}{\partial x}\right)^2. \tag{23}$$

We substitute equality (23) into (22), multiply by τ and sum the resulting relation over t from 0 to $t' - \tau$ and obtain

$$\frac{1}{2}\|u^n(t')\|_1^2 - \frac{1}{2}\|u^n(0)\|_1^2 + \frac{1}{2}\sum_{t=0}^{t'-\tau}\tau^2\|u_t^n(t)\|_1^2 + \sum_{t=0}^{t'-\tau}\tau\left\|\frac{\partial u_t^n}{\partial x}\right\|_{L_2(0,L)}^2 +$$

$$+\sum_{t=0}^{t'-\tau}\tau\int_0^L g\left(\left|\frac{\partial\hat{p}^n}{\partial x}\right|\right)\left(\frac{\partial\hat{p}^n}{\partial x}\right)^2 dx=\sum_{t=0}^{t'-\tau}\tau\int_0^L\hat{f}(x,t)\cdot u_t^n dx. \quad (24)$$

From (24), taking into account inequality (6), we have a priori estimates (19)–(20). Also, considering that

$$\left\|g\left(\left|\frac{\partial\hat{p}^n}{\partial x}\right|\right)\frac{\partial\hat{p}^n}{\partial x}\right\|^2_{L_2(0,L)}\le\left\|\frac{\partial\hat{p}^n}{\partial x}\right\|^2_{L_2(0,L)}.$$

we have estimate (21). The proof of Lemma 1 is complete.

Lemma 2. There exist function

$$u\in W_2^{(1)}(0,T;\overset{\circ}{V}),\qquad p\in L_2(0,T;\overset{\circ}{V}_1)$$

and sequences $\{\tau\},\{n\}$ such that at $\tau\longrightarrow 0,\quad n\longrightarrow\infty$

$$\Pi^+u^n\rightharpoonup u,\quad \Pi^+u_t^n\rightharpoonup\frac{\partial u}{\partial t}\quad\text{in } L_2(0,T;\overset{\circ}{V}), \quad (25)$$

$$\frac{\partial\Pi^+u_t^n}{\partial x}\rightharpoonup\frac{\partial^2 u}{\partial x\partial t}\quad\text{in } L_2(0,T;L_2(0,L)), \quad (26)$$

$$\Pi^+p^n\rightharpoonup p\ \text{ in } L_2(0,T;\overset{\circ}{V}_1). \quad (27)$$

Here Π^+z is piecewise-constant filling of z :

$$\Pi^+z(t)=\left\{z(k\tau)\ :\ k\tau\le t<(k+1)\tau\right\}.$$

The validity of statements (25)–(27) follows from a priori estimates (19)–(20) and the weak compactness of bounded sets in a reflexive Banach space. The proof of Lemma 2 is complete.

Theorem 2. Functions u, p satisfying relations (25)–(27) are a generalized solution to problem (1)–(5).

Proof. Let the functions u, p satisfy relations (25)–(27), it is required to prove that u, p satisfy identity (7). To do this, in (8) we put

$$v^n(x,t)=\frac{1}{\tau}\int_t^{t+\tau}\tilde{v}^n(x,\xi)d\xi,\quad z^n(x,t)=\frac{1}{\tau}\int_t^{t+\tau}\tilde{z}^n(x,\xi)d\xi,$$

where $\tilde{v}^n$, $\tilde{z}^n$ are functions from $C^\infty(0,T;\overset{\circ}{V}{}^n)$ and $C^\infty(0,T;\overset{\circ}{V}{}_1^n)$ respectively, such that $\tilde{v}^n(x,T)=\tilde{z}^n(x,T)=0$. We multiply (8) by τ, sum over t from 0 to $T-\tau$. The result, using the filling operator Π^+, can be written in the form

$$\int_0^T\int_0^L\left\{\left(\frac{\partial\Pi^+\hat{u}^n}{\partial x}+\frac{\partial\Pi^+u_t^n}{\partial x}\right)\frac{\partial\Pi^+v_t^n}{\partial x}-\Pi^+\hat{p}^n\frac{\partial\Pi^+v_t^n}{\partial x}+\frac{\partial\Pi^+u_t^n}{\partial x}\Pi^+\hat{z}^n+\right.$$
$$\left.+g\left(\left|\frac{\partial\Pi^+\hat{p}^n}{\partial x}\right|\right)\frac{\partial\Pi^+\hat{p}^n}{\partial x}\frac{\partial\Pi^+\hat{z}^n}{\partial x}\right\}dxdt=\int_0^T\int_0^L\hat{f}(x,t)\,\Pi^+v_t^n dxdt. \quad (28)$$

From the boundedness of g and estimate (21) it follows that there exists a function χ from the space $L_2(0, T; L_2(0, L))$ such that

$$g\left(\left|\frac{\partial \Pi^+ \hat{p}^n}{\partial x}\right|\right)\frac{\partial \Pi^+ \hat{p}^n}{\partial x} \rightharpoonup \chi \qquad \text{in} \quad L_2(0, T; L_2(0, L)). \tag{29}$$

Taking into account (25)–(27) and (29) in equality (28), we pass to the limit in $\tau \longrightarrow 0$ and $n \longrightarrow \infty$ and obtain

$$\int_0^T \int_0^L \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t}\right)\frac{\partial^2 v}{\partial x \partial t} - p\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} z + \chi \frac{\partial z}{\partial x}\right\} dxdt =$$

$$= \int_0^T \int_0^L f(x,t)\frac{\partial v}{\partial t} dxdt. \tag{30}$$

Let us prove that $\chi = g\left(\left|\frac{\partial p}{\partial x}\right|\right)\frac{\partial p}{\partial x}$. To do this, we use the monotonicity method. We write down the apparent inequality

$$\sum_{t=0}^{T-\tau} \tau \int_0^L \left(\frac{\partial u^n}{\partial x} - \frac{\partial v^n}{\partial x}\right)_t \left(\frac{\partial \hat{u}^n}{\partial x} - \frac{\partial \hat{v}^n}{\partial x}\right) dx \geq$$

$$\geq \frac{1}{2}\|u^n(T) - v^n(T)\|_1^2 \frac{\partial^2 u}{\partial x \partial t} z - \frac{1}{2}\|u^n(0) - v^n(0)\|_1^2 \geq -\frac{1}{2}\|u_0 - v^n(x,0)\|_1^2,$$

where v^n is an arbitrary smooth function $v \in C^\infty(0, T; \overset{\circ}{V}{}^n)$. From this inequality and the monotonicity of the function $g(\xi)$ it follows that

$$\sum_{t=0}^{T-\tau} \tau \int_0^L \left(\frac{\partial u^n}{\partial x} - \frac{\partial v^n}{\partial x}\right)_t \left(\frac{\partial \hat{u}^n}{\partial x} - \frac{\partial \hat{v}^n}{\partial x}\right) dx +$$

$$+ \sum_{t=0}^{T-\tau} \tau \int_0^L \left\{ g\left(\left|\frac{\partial \hat{p}^n}{\partial x}\right|\right)\frac{\partial \hat{p}^n}{\partial x} - g\left(\left|\frac{\partial \hat{z}^n}{\partial x}\right|\right)\frac{\partial \hat{z}^n}{\partial x}\right\} \frac{\partial\left(\hat{p}^n - \hat{z}^n\right)}{\partial x} dx \geq -\frac{1}{2}\|u_0 - v^n(x,0)\|_1^2.$$

The last relation is equivalent to the following integral inequality

$$I_{\tau,n} = \int_0^T \int_0^L \left(\frac{\partial \Pi^+ u_t^n}{\partial x} - \frac{\partial \Pi^+ v_t^n}{\partial x}\right)\frac{\partial \Pi^+\left(\hat{u}^n - \hat{v}^n\right)}{\partial x} dxdt +$$

$$+ \int_0^T \int_0^L g\left(\left|\frac{\partial \Pi^+ \hat{p}^n}{\partial x}\right|\right)\frac{\partial \Pi^+ \hat{p}^n}{\partial x}\frac{\partial \Pi^+\left(\hat{p}^n - \hat{z}^n\right)}{\partial x} dxdt -$$

$$- \int_0^T \int_0^L g\left(\left|\frac{\partial \Pi^+ \hat{z}^n}{\partial x}\right|\right)\frac{\partial \Pi^+ \hat{z}^n}{\partial x}\frac{\partial \Pi^+\left(\hat{p}^n - \hat{z}^n\right)}{\partial x} dxdt \geq -\frac{1}{2}\|u_0 - v^n(x,0)\|_1^2. \tag{31}$$

We represent $I_{\tau,n}$ as the sum $I = I^{(1)}_{\tau,n} + I^{(2)}_{\tau,n}$, where

$$I^{(1)}_{\tau,n} = \int\limits_0^T \int\limits_0^L \left\{ \frac{\partial \Pi^+ u_t^n}{\partial x} \frac{\partial \Pi^+ \left(\hat{u}^n - \hat{v}^n\right)}{\partial x} + g\left(\left| \frac{\partial \Pi^+ \hat{p}^n}{\partial x} \right| \right) \frac{\partial \Pi^+ \hat{p}^n}{\partial x} \frac{\partial \Pi^+ \left(\hat{p}^n - \hat{z}^n\right)}{\partial x} \right\} dxdt,$$

$$I^{(2)}_{\tau,n} = -\int\limits_0^T \int\limits_0^L \left\{ \frac{\partial \Pi^+ v_t^n}{\partial x} \frac{\partial \Pi^+ \left(\hat{u}^n - \hat{v}^n\right)}{\partial x} + g\left(\left| \frac{\partial \Pi^+ \hat{z}^n}{\partial x} \right| \right) \frac{\partial \Pi^+ \hat{z}^n}{\partial x} \frac{\partial \Pi^+ \left(\hat{p}^n - \hat{z}^n\right)}{\partial x} \right\} dxdt.$$

To transform the first relation $I^{(1)}_{\tau,n}$, we use equality (28) at $v^n = u^n - v^n$, $p^n = p^n - z^n$ and obtain

$$I^{(1)}_{\tau,n} = \int\limits_0^T \int\limits_0^L \left\{ -\frac{\partial \Pi^+ u_t^n}{\partial x} \frac{\partial \Pi^+ \left(u_t^n - v_t^n\right)}{\partial x} - \Pi^+ \hat{p}^n \frac{\partial \Pi^+ v_t^n}{\partial x} + \frac{\partial \Pi^+ u_t^n}{\partial x} \Pi^+ \hat{z}^n - \frac{\partial \Pi^+ u_t^n}{\partial x} \frac{\partial \Pi^+ v^n}{\partial x} + \right.$$

$$\left. + \frac{\partial \Pi^+ u^n}{\partial x} \frac{\partial \Pi^+ v_t^n}{\partial x} + \hat{f}(x,t)\, \Pi^+ \left(u^n - v^n\right)_t \right\} dxdt. \quad (32)$$

In (32)), we make the passage to the limit as $\tau \longrightarrow 0$, $n \longrightarrow \infty$, taking into account (25)–(27), (29). As a result, we obtain

$$I^{(1)}_{\tau,n} \longrightarrow \int\limits_0^T \int\limits_0^L \left\{ -\frac{\partial^2 u}{\partial x \partial t} \frac{\partial^2 (u - v)}{\partial x \partial t} - p \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} z - \right.$$

$$\left. - \frac{\partial^2 u}{\partial x \partial t} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t} + f(x,t) \frac{\partial (u - v)}{\partial t} \right\} dxdt. \quad (33)$$

Using equality (30), the right-hand side of relation (33) takes the following form

$$I^{(1)}_{\tau,n} \longrightarrow \int\limits_0^T \int\limits_0^L \left\{ \frac{\partial^2 u}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + \chi \frac{\partial (p - z)}{\partial x} \right\} dxdt. \quad (34)$$

Apparently, from (25)–(27), (29) for $\tau \longrightarrow 0$, $n \longrightarrow \infty$ we obtain

$$I^{(2)}_{\tau,n} \longrightarrow -\int\limits_0^T \int\limits_0^L \left\{ \frac{\partial^2 v}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + g\left(\left| \frac{\partial z}{\partial x} \right| \right) \frac{\partial z}{\partial x} \frac{\partial (p - z)}{\partial x} \right\} dxdt. \quad (35)$$

Thus, it follows from the definition of $I_{\tau,n}$ that

$$\int\limits_0^T \int\limits_0^L \left\{ \frac{\partial^2 (u - v)}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + \left(\chi - g\left(\left| \frac{\partial z}{\partial x} \right| \right) \frac{\partial z}{\partial x} \right) \frac{\partial (p - z)}{\partial x} \right\} dxdt \geq$$

$$\geq -\frac{1}{2} \| u_0 - v(x,0) \|_1^2. \quad (36)$$

In (36), we choose $v = u + \lambda w$, $z = p + \lambda q$, where $\lambda = \text{const} > 0$, and w, q are arbitrary functions from $C^\infty(0,T; C^\infty(0,L))$, where $w(x,0) = 0$ for $x \in (0,L)$. As a result, we obtain

$$\lambda \int_0^T \int_0^L \left(\chi - g\left(\left| \frac{\partial (p+\lambda q)}{\partial x} \right| \right) \frac{\partial (p+\lambda q)}{\partial x} \right) \frac{\partial q}{\partial x} dx dt +$$

$$+ \lambda^2 \int_0^T \int_0^L \frac{\partial^2 w}{\partial x \partial t} \frac{\partial w}{\partial x} dx dt \geq -\frac{\lambda}{2} \| w(x,0) \|_1^2 = 0. \quad (37)$$

We divide inequality (37) by λ and pass to the limit as $\lambda \longrightarrow 0$, we obtain

$$\int_0^T \int_0^L \left(\chi - g\left(\left| \frac{\partial p}{\partial x} \right| \right) \frac{\partial p}{\partial x} \right) \frac{\partial q}{\partial x} dx dt \geq 0. \quad (38)$$

Since q is an arbitrary function, the inequality holds at $q = v$ and $q = -v$, where $v \in L_2(0, T; W_2^1(0, L))$ is an arbitrary function; therefore, we have

$$\chi = g\left(\left| \frac{\partial p}{\partial x} \right| \right) \frac{\partial p}{\partial x}.$$

The proof of theorem 2 is complete.

References

[1] F. Kadyrov, A. Kosterin, E. Skvorcov, Plane problem of filtration consolidation for an elastic half-space with discontinuous initial conditions, Prikl. mekh. i tekhn. fiz. 57 (2016) 1–7.

[2] Y. Zareckij, Theory of soil consolidation, Nauka, Moscow, 2001.

[3] V. Nikolaevskij, Mechanics of porous and fractured media, Nedra, Moscow, 1984.

[4] M. Biot, The mechanics of deformation and acoustic propagation in porous media, J. Appl. Phys. 33 (1962) 1482–1498.

[5] A. Egorov, A. Kosterin, E. Skvorcov, Consolidation and acoustic waves in saturated porous media, KSU Publishing House, Kazan, 1990.

[6] M. Pavlova, E. Rung, On the solvability of the problem of saturated-unsaturated filtration consolidation, Diff. Eq. 48 (2012) 990–1004.

[7] M. Pavlova, E. Rung, On the existence of a generalized solution of the saturated-unsaturated filtration problem, Diff. Eq. 54 (2018) 352–362.

[8] H.-J. G. Diersch, P. Perrochet, On the primary variable switching technique for simulating unsaturated-saturated flows, Advances in Water Resources 23 (1999) 271–301.

[9] A. Akhtareev, R. Dautov, Mixed variable method for modeling saturated-unsaturated flows, Uchen. Zap. Kazan Gos. Univ. Fiz-Mat. Nauki 149 (2007) 58–72.

[10] G. Williams, C. Miller, C. Kelley, Transformation approaches for simulating flow in variably saturated porous media, Water Resources Research 36 (2000) 923–934.

[11] J.-L. Lions, Quelques méthodes de résolution des problémes aux limites nonlinéaires, Dunod, Paris, 1969.