# The general algorithm of linearization in linear fractional optimization problems

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**Abstract**

One of the most common examples of using the linear fractional optimization in project management is given by a problem of minimizing the expense per a unit of time or resource while maximizing the tasks completion quality. For example, in planning of a construction project, managers can use linear fractional models for optimizing the expense for construction materials and manpower with ensuring a high quality of works and good meeting of the schedule milestones at the same time.

**Keywords**

linear fractional optimization, project, resources, risk management

## 1. Introduction

In the rapidly developing world of nowadays, information technologies become a foundation for solving complicated problems in various branches of science and industry. The optimization is one of such important areas that enables finding the best solutions for attaining results desired while minimizing the expense or maximizing the efficiency.

Problems of linear fractional optimization appear in many real situations when it is required to optimize the ratio between multiple parameters. For instance, this can be a problem of minimizing the prime cost of a product subject to reserving a certain level of quality or maximizing the profit when resources are restricted. Such problems are often of a nonlinear origin, which complicates their solution aided by conventional linear programing methods. Therefore, one of important areas of research consists in linearization of such problems that enables using efficient linear optimization algorithms.

## 2. Statement of basic material

The linear fractional optimization in project management is one of the most efficient approaches for solving complicated problems associated with planning, distribution of resources and management of risks. This method enables optimizing the ratio between multiple important parameters of a project, such as value, time, execution quality and other indicators being of critical importance for successful project closeout.

The linear fractional optimization differs from conventional methods in the fact that the objective function in such problems is written as a fraction where the numerator and the denominator are linear functions. For instance, it can be a ratio of expense vs executed works quality or of the project fulfilment time vs the project quality. Problems of this type frequently appear in the real life,

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especially in the project management where it is essential to find the optimum ratio between various resources and results.

One of the most common examples of using the linear fractional optimization in project management is given by a problem of minimizing the expense per a unit of time or resource while maximizing the tasks completion quality. For example, in planning of a construction project, managers can use linear fractional models for optimizing the expense for construction materials and manpower with ensuring a high quality of works and good meeting of the schedule milestones at the same time.

Another example is given by resources usage optimization. In large-scale projects, such as construction of infrastructures or implementation of new technologies, it is required to efficiently distribute the resources across multiple project stages. Using the linear fractional optimization, you can minimize the expense per one unit of productivity or increase the productivity within a constricted budget, which is essential for a successful project closeout.

As it is difficult to solve linear fractional problems with the help of conventional optimization methods, they are frequently linearized, i.e. converted into linear form. This is attained by introducing new variables that allow reducing the fractional function to a linear one, after which standard linear programing methods, such as simplex method, can be applied.

The linear fractional optimization is also used in risk management. In the project management, risks are often measured in the form of a ratio of a certain event probability vs its consequences. Linear fractional optimization can help to minimize potential expenses with maximizing the efficiency of risk management measures at the same time.

The advantages of linear fractional optimization include the possibility of taking account of complicated relationship between multiple project parameters, which enables obtaining more accurate and well-balanced solutions. It also provides flexibility in decision taking, as the managers can model various event development scenarios and select the most optimum way.

However, it is worth mentioning that the linear fractional optimization is the most complicated from the calculation point of view. It requires significant resources for solving and may require special algorithms and software. This creates a certain challenge for its utilization in real projects, especially in cases with a big number of variables and constraints.

## 3. Problem research

Mathematic models of mixed project management optimization often use nonlinear functions like:

$$\mathrm{W_I} == \frac{P_1(x_1, x_2, \ldots, x_n)}{Q_1(x_1, x_2, \ldots, x_n)} = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j},$$

where

$$x_j \geq 0, \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$$

$$c_j, d_j - \text{const}, \quad \sum_{j=1}^{n} d_j x_j \neq 0.$$

Similar nonlinear objective functions are used for mathematic models of economic specialization:

- The objective function of a model for optimizing the product manufacture expense profitability:

$$W_I == \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j} \to \max$$
,

where

$x_j$ - the quantity of product planned to manufacture,

$c_j$ - the profit from selling one unit of product $x_j$,

$d_j$ - the prime cost of producing one unit of product $x_j$.

- The objective function of a model for optimizing the product sales profitability:

$$W_I == \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j} \to \max$$
,

where

$x_j$ - the quantity of product planned to sales,

$c_j$ - the profit from selling one unit of product $x_j$,

$d_j$ - the price of one unit of product $x_j$.

- The objective function of a model for optimizing the expense per one monetary unit of product:

$$W_I == \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j} \to \min$$
,

where

$x_j$ - the quantity of product planned to sales,

$c_j$ - the prime cost of one product unit manufacture $x_j$,

$d_j$ - the price of one unit of product $x_j$.

- The objective function of a model for optimizing the product manufacture prime cost:

$$W_I == \frac{\sum_{j=1}^{n} d_j}{\sum_{j=1}^{n} x_j} \to \min$$
,

where

$x_j$ - the quantity of product produced,

$d_j$ - the price of one unit of product $x_j$.

In view of this, it is important to linearize objective functions of models to reduce the mathematic model to a linear optimization problem.

| Basis | C | B | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 3 | –2 | 0 | 4 | 0 | 0 |
| $a_5$ | 0 | –3 | –8 | –2 | –4 | 0 | 1 | 1 | 0 |
| $a_6$ | 0 | 5 | 4 | 7 | 9 | 0 | 4 | 0 | 1 |
| $a_3$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $\Delta_j$ | $W_I(Z_0) =$ | –1 | –1 | –3 | 2 | 0 | –4 | 0 | 0 |
| $a_0$ | 1 | 3/8 | 1 | 1/4 | 1/2 | 0 | – 1/8 | – 1/8 | 0 |
| $a_6$ | 0 | 7/2 | 0 | 6 | 7 | 0 | 9/2 | 1/2 | 1 |
| $a_3$ | 0 | 5/8 | 0 | 3/4 | 1/2 | 1 | 9/8 | 1/8 | 0 |
| $\Delta_j$ | $W_I(Z_1) =$ | – 5/8 | 0 | –11/4 | 5/2 | 0 | –33/8 | – 1/8 | 0 |
| $a_0$ | 1 | 4/9 | 1 | 1/3 | 5/9 | 1/9 | 0 | – 1/9 | 0 |
| $a_6$ | 0 | 1 | 0 | 3 | 5 | –4 | 0 | 0 | 1 |
| $a_4$ | 4 | 5/9 | 0 | 2/3 | 4/9 | 8/9 | 1 | 1/9 | 0 |
| $\Delta_j$ | $W_I(Z_2) =$ | 5/3 | 0 | 0 | 13/3 | 11/3 | 0 | 1/3 | 0 |

It is known that such an optimization problem is called linear fractional optimization problem where the objective function is a linear fractional function and the system of constraints complies with conditions of linearity, i.e. they are linear equations or inequalities.

A general linear fractional optimization problem shall look like:

$$
\begin{gathered}
\mathrm{W}_\mathrm{I} = \frac{P_1(x_1, x_2, \ldots, x_n)}{Q_1(x_1, x_2, \ldots, x_n)} = \frac{\sum\limits_{j=1}^{n} c_j x_j}{\sum\limits_{j=1}^{n} d_j x_j} \to \mathrm{opt}(\max, \min), \\
\Omega_\mathrm{I} : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases} \\
x_j \geq 0, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \\
c_j, d_j \, b_i \, a_{ij} - \text{const}, \quad \sum_{j=1}^{n} d_j x_j \neq 0.
\end{gathered}
\tag{1}
$$

We reduce linear fractional optimization problem (1) to solution of a linear optimization problem.

Let us designate $z_0 = \dfrac{1}{\sum\limits_{j=1}^{n} d_j x_j}$, and introduce new variables: $z_j = z_0 x_j \;\; j = 1, 2, \ldots, n$.

Problem (1) turns to:

$$W_{Iz} = \sum_{j=1}^{n} c_j z_j \to \text{opt}(\max, \min),$$

$$\left[\begin{array}{l} \Omega_{Iz}: \begin{cases} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n - b_1 z_0 = 0, \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n - b_2 z_0 = 0, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}z_1 + a_{m2}z_2 + \cdots + a_{mn}z_n - b_m z_0 = 0, \end{cases} \\ \quad d_1 z_1 + d_2 z_2 + \cdots + d_n z_n = 1, \end{array}\right.$$

$$z_j \geq 0,\ z_0 > 0, i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n. \tag{2}$$

Additional problem (2) is a set of two problems. The first problem is a linear optimization problem. Therefore, it is solved by simplex method and then we are to find a solution to the initial linear fractional problem. The second problem is associated with designation $z_0 = \dfrac{1}{\sum_{j=1}^{n} d_j x_j}$ for simplifying the problem solution in general.

***Model example No.* 1**.

Let us find the solution of problem

$$W_I = \frac{2x_1 + 3x_2 - x_3}{x_1 + 2x_2 + x_3 + 3} \to \text{opt}(\max),$$

$$\Omega_I: \begin{cases} x_1 + 2x_2 + x_3 \leq 4, \\ -5x_1 - x_2 - 3x_3 \leq 1, \end{cases}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \tag{3}$$

We have a linear fractional optimization problem. In the system of constraints, we move from inequality constraints to equation constraints:

$$\Omega_I: \begin{cases} x_1 + 2x_2 + x_3 \leq 4, \\ -5x_1 - x_2 - 3x_3 \leq 1, \end{cases} \Rightarrow \Omega_I: \begin{cases} x_1 + 2x_2 + x_3 + x_4 = 4, \\ -5x_1 - x_2 - 3x_3 + x_5 = 1, \end{cases}$$

$$x_j \geq 0, j = 1, 2, \ldots, 5. \tag{4}$$

Let us designate

$$z_0 = \frac{1}{x_1 + 2x_2 + x_3 + 3},$$

and introduce new variables:

$$z_1 = z_0 x_1,\ z_2 = z_0 x_2, \ldots,\ z_5 = z_0 x_5. \tag{5}$$

The objective function of additional problem (2) turns to:

$$W_{Iz} = 2z_1 + 3z_2 - z_3 \to \text{opt}(\max, \min).$$

We shell multiply both the system of constraints equation members (4) by $z_0$ and move to new variables (5).

Additional problem (2) turns to:

$$W_{Iz} = 2z_1 + 3z_2 - z_3 \to \text{opt}(\max, \min)$$

$$\left[\begin{array}{l} \Omega_{Iz}: \begin{cases} -4z_0 + z_1 + 2z_2 + z_3 + z_4 = 0, \\ -z_0 - 5z_1 - z_2 - 3z_3 + z_5 = 0, \end{cases} \\ \quad 3z_0 + z_1 + 2z_2 + z_3 = 1, \end{array}\right.$$

$$z_j \geq 0, z_0 > 0, j = 1, 2, \ldots, 5. \tag{6}$$

It is known that the beginning of solving a linear optimization problem with simplex method requires obtaining the primary basis $Z_0$ with necessary use of the Jordan-Gauss complete elimination method (Table 1).

**Table 1**

A linear optimization problem with simplex method

| | $z_0$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $b$ | $\Sigma$ |
|---|---|---|---|---|---|---|---|---|
| | –4 | 1 | 2 | 1 | 1 | 0 | 0 | 1 |
| | –1 | –5 | –1 | –3 | 0 | 1 | 0 | –9 |
| | 3 | 1 | 2 | 1 | 0 | 0 | 1 | 8 |
| $W_{iz}$ | 0 | 2 | 3 | –1 | 0 | 0 | 0 | |
| | –7 | 0 | 0 | 0 | 1 | 0 | –1 | –7 |
| | 8 | –2 | 5 | 0 | 0 | 1 | 3 | 15 |
| | 3 | 1 | 2 | 1 | 0 | 0 | 1 | 8 |
| $W_{iz}$ | 3 | 3 | 5 | 0 | 0 | 0 | 1 | |

We have $Z_0 = [0, 0, 0, 8, -7, 15]$.

Table 2 provides a simplex calculation of additional problem (6).

**Table 2**

A simplex calculation of additional problem (6)

| *Basis* | *C* | *B* | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|---|---|
| | | | 3 | 3 | 0 | 0 | 0 | 0 |
| $a_4$ | 0 | –1 | –7 | 0 | 0 | 0 | 1 | 0 |
| $a_5$ | 0 | 3 | 8 | –2 | 5 | 0 | 0 | 1 |
| $a_3$ | 0 | 1 | 3 | 1 | 2 | 1 | 0 | 0 |
| $\Delta_j$ | $W_1(Z_0) = -1$ | | –3 | –3 | 0 | 0 | 0 | 0 |
| $a_0$ | 3 | 1/7 | 1 | 0 | 0 | 0 | – 1/7 | 0 |
| $a_5$ | 0 | 13/7 | 0 | –2 | 5 | 0 | 8/7 | 1 |
| $a_3$ | 0 | 4/7 | 0 | 1 | 2 | 1 | 3/7 | 0 |
| $\Delta_j$ | $W_1(Z_1) = -4/7$ | | 0 | –3 | 0 | 0 | – 3/7 | 0 |
| $a_0$ | 3 | 1/7 | 1 | 0 | 0 | 0 | – 1/7 | 0 |
| $a_5$ | 0 | 3 | 0 | 0 | 9 | 2 | 2 | 1 |
| $a_1$ | 3 | 4/7 | 0 | 1 | 2 | 1 | 3/7 | 0 |
| $\Delta_j$ | $W_1(Z_2) = 8/7$ | | 0 | 0 | 6 | 3 | 6/7 | 0 |

The optimum solution of additional problem (6) is equal to:

$$Z_{\text{opt}} = Z_2 = \left[\frac{1}{7}, \frac{4}{7}, 0, 0, 0, 3\right],$$

then

$$x_1^{\text{opt}} = \frac{z_1^{\text{opt}}}{z_0} = \frac{\frac{4}{7}}{\frac{1}{7}} = 4, \quad x_2^{\text{opt}} = \frac{z_2^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{7}} = 0, \quad x_3^{\text{opt}} = \frac{z_3^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{7}} = 0,$$

$$x_4^{\text{opt}} = \frac{z_4^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{7}} = 0, \quad x_5^{\text{opt}} = \frac{z_5^{\text{opt}}}{z_0} = \frac{3}{\frac{1}{7}} = 21.$$

For obtaining a solution to the minimization problem, it is worth mentioning that the current basis $Z_1$ of solving the maximization problem with simplex method (Table No. 2) is the solution to the minimization problem as all estimates in the simplex table are nonpositive.

We finally have: $X_{\text{opt}}^{\max} = [4,0,0]$, $\mathrm{W}_{\mathrm{I}}^{\max} = \frac{8}{7}$, $X_{\text{opt}}^{\min} = [0,0,4]$, $\mathrm{W}_{\mathrm{I}}^{\min} = -\frac{4}{23}$.

***Model example No.*** 2.

We need to find the solution to problem

$$\mathrm{W}_{\mathrm{I}} = \frac{2x_1 - 3x_2 - x_3 + 3x_4}{x_1 + x_2 + x_3 + x_4 + 1} \to \text{opt}(\max, \min),$$

$$\Omega_{\mathrm{I}} : \begin{cases} x_1 - x_2 + 3x_3 + 4x_4 \le 5, \\ 2x_1 + 4x_2 - 5x_3 - x_4 \le 1, \end{cases}$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0. \tag{7}$$

We have a linear fractional optimization problem. In the system of constraints, we move from inequality constraints to equation constraints:

$$\Omega_{\mathrm{I}} : \begin{cases} x_1 - x_2 + 3x_3 + 4x_4 \le 5, \\ 2x_1 + 4x_2 - 5x_3 - x_4 \le 1, \end{cases} \Rightarrow \Omega_{\mathrm{I}} : \begin{cases} x_1 - x_2 + 3x_3 + 4x_4 + x_5 = 5, \\ 2x_1 + 4x_2 - 5x_3 - x_4 + x_6 = 1, \end{cases}$$

$$x_j \ge 0, j = 1, 2, \ldots, 6. \tag{8}$$

We designate

$$z_0 = \frac{1}{x_1 + x_2 + x_3 + x_4 + 1},$$

and introduce new variables:

$$z_1 = z_0 x_1, \; z_2 = z_0 x_2, \ldots, z_5 = z_0 x_5, \; z_6 = z_0 x_6 \tag{9}$$

In this case, the objective function of additional problem (2) turns to:

$$\mathrm{W}_{\mathrm{I}z} = 2z_1 - 3z_2 - z_3 + 3z_4 \to \text{opt}(\max, \min).$$

We shall multiply both the system of constraints equation members (4) by $z_0$ and move to new variables (5).

Additional problem (2) turns to:

$$\mathrm{W}_{1z} = 2z_1 + 3z_2 - z_3 + 3z_4 \to \mathrm{opt}(\max, \min)$$

$$\left[\begin{array}{l} \Omega_{1z} : \begin{cases} -5z_0 + z_1 - z_2 + 3z_3 + 4z_4 + z_5 = 0, \\ -z_0 + 2z_1 + 4z_2 - 5z_3 - z_4 + z_6 = 0, \end{cases} \\ \quad z_0 + z_1 + z_2 + z_3 + z_4 = 1, \end{array}\right.$$

$$z_j \geq 0,\, z_0 > 0,\, j = 1, 2, \ldots, 6. \tag{10}$$

It is known that the beginning of solution to a linear optimization problem with simplex method requires obtaining the primary basis $Z_0$ with necessary use of the Jordan-Gauss complete elimination method (Table 3).

**Table 3**
A linear optimization problem with simplex method use of the Jordan-Gauss complete elimination method

| | $z_0$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $b$ | $\Sigma$ |
|---|---|---|---|---|---|---|---|---|---|
| | –5 | 1 | –1 | 3 | 4 | 1 | 0 | 0 | 3 |
| | –1 | 2 | 4 | –5 | –1 | 0 | 1 | 0 | 0 |
| | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 6 |
| $\mathrm{W}_{iz}$ | 0 | 2 | –3 | –1 | 3 | 0 | 0 | 0 | |
| | | | | | | | | | |
| | –8 | –2 | –4 | 0 | 1 | 1 | 0 | –3 | –15 |
| | 4 | 7 | 9 | 0 | 4 | 0 | 1 | 5 | 30 |
| | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 6 |
| $\mathrm{W}_{iz}$ | 1 | 3 | –2 | 0 | 4 | 0 | 0 | 1 | |

We have $Z_0 = [0, 0, 0, 1, 0, -3, 5]$.

The optimum solution to additional problem (6) is equal to:

$$Z_{\mathrm{opt}} = Z_2 = \left[\frac{4}{9}, 0, 0, 0, \frac{5}{9}, 0, 1\right],$$

then

$$x_1^{\mathrm{opt}} = \frac{z_1^{\mathrm{opt}}}{z_0} = \frac{0}{\frac{4}{9}} = 0, \quad x_2^{\mathrm{opt}} = \frac{z_2^{\mathrm{opt}}}{z_0} = \frac{0}{\frac{4}{9}} = 0, \quad x_3^{\mathrm{opt}} = \frac{z_3^{\mathrm{opt}}}{z_0} = \frac{0}{\frac{4}{9}} = 0,$$

$$x_4^{\mathrm{opt}} = \frac{z_4^{\mathrm{opt}}}{z_0} = \frac{\frac{5}{9}}{\frac{4}{9}} = \frac{5}{4}, \quad x_5^{\mathrm{opt}} = \frac{z_5^{\mathrm{opt}}}{z_0} = \frac{0}{\frac{4}{9}} = 0, \quad x_6^{\mathrm{opt}} = \frac{z_6^{\mathrm{opt}}}{z_0} = \frac{1}{\frac{4}{9}} = \frac{9}{4}.$$

Table 4 provides a simplex calculation to additional minimization problem (6).

**Table 4**
A simplex calculation to additional minimization problem (6).

| Basis | C | B | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 3 | –2 | 0 | 4 | 0 | 0 |
| $a_5$ | 0 | –3 | –8 | –2 | –4 | 0 | 1 | 1 | 0 |
| $a_6$ | 0 | 5 | 4 | 7 | 9 | 0 | 4 | 0 | 1 |
| $a_3$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $\Delta_j$ | $W_I(Z_0) = -1$ | | –1 | –3 | 2 | 0 | –4 | 0 | 0 |
| $a_0$ | 1 | 3/8 | 1 | 1/4 | 1/2 | 0 | – 1/8 | – 1/8 | 0 |
| $a_6$ | 0 | 7/2 | 0 | 6 | 7 | 0 | 9/2 | 1/2 | 1 |
| $a_3$ | 0 | 5/8 | 0 | 3/4 | 1/2 | 1 | 9/8 | 1/8 | 0 |
| $\Delta_j$ | $W_I(Z_1) = -5/8$ | | 0 | –11/4 | 5/2 | 0 | –33/8 | – 1/8 | 0 |
| $a_0$ | 1 | 1/8 | 1 | – 5/28 | 0 | 0 | –25/56 | – 9/56 | – 1/14 |
| $a_2$ | –2 | 1/2 | 0 | 6/7 | 1 | 0 | 9/14 | 1/14 | 1/7 |
| $a_3$ | 0 | 3/8 | 0 | 9/28 | 0 | 1 | 45/56 | 5/56 | – 1/14 |
| $\Delta_j$ | $W_I(Z_2) = -15/8$ | | 0 | –137/28 | 0 | 0 | –321/56 | –17/56 | – 5/14 |

The optimum solution to additional minimization problem (6) is equal to:

$$Z_{\min} = Z_2 = \left[\frac{1}{8}, 0, \frac{1}{2}, \frac{3}{8}, 0, 0, 0\right],$$

then

$$x_1^{\text{opt}} = \frac{z_1^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{8}} = 0, \quad x_2^{\text{opt}} = \frac{z_2^{\text{opt}}}{z_0} = \frac{\frac{1}{2}}{\frac{1}{8}} = 4, \quad x_3^{\text{opt}} = \frac{z_3^{\text{opt}}}{z_0} = \frac{\frac{3}{8}}{\frac{1}{8}} = 3,$$

$$x_4^{\text{opt}} = \frac{z_4^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{8}} = 0, \quad x_5^{\text{opt}} = \frac{z_5^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{8}} = 0, \quad x_6^{\text{opt}} = \frac{z_6^{\text{opt}}}{z_0} = \frac{0}{\frac{1}{8}} = 0.$$

We have: $X_{\text{opt}}^{\max} = \left[0, 0, 0, \frac{5}{4}\right]$, $W_I^{\max} = \frac{5}{3}$, $X_{\text{opt}}^{\min} = [0, 4, 3, 0]$, $W_I^{\min} = -\frac{15}{8}$.

In a two-dimensional case, a linear fractional optimization problem can be graphically solved with graphic interpretation of the solution.

A linear fractional problem of two variables optimization shall be formulated as follows: we need to fine such a basis $X_{\text{opt}} = [x_1, x_2]$ that provides the optimum value to objective function $W_I(X_{\text{opt}}) = W_I^{\text{opt}}$

$$W_I = \frac{P_2(x_1,x_2)}{Q_2(x_1,x_2)} = \frac{\sum_{j=1}^{2} c_j x_j}{\sum_{j=1}^{2} d_j x_j} \to \text{opt}(\max,\min),$$

$$\Omega_I : \begin{cases} a_{11}x_1 + a_{12}x_2 \le b_1, \\ a_{21}x_1 + a_{22}x_2 \le b_2, \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 \le b_m, \end{cases}$$

$$x_j \ge 0, \quad i = 1,2,\dots,m, \; j = 1,2,$$

$$c_j, d_j \; b_i \; a_{ij} - \text{const}, \quad \sum_{j=1}^{2} d_j x_j \ne 0. \tag{11}$$

Let us consider the solution and the geometric interpretation of the problem solution with two variables. Two cases are possible:

- Objective function of the problem is a homogeneous function like:

$$W_I = \frac{P_2(x_1,x_2)}{Q_2(x_1,x_2)} = \frac{c_1x_1 + c_2x_2}{d_1x_1 + d_2x_2} \to \text{opt}(\max,\min), \tag{12}$$

- Objective function of the problem is a nonhomogeneous function like:

$$W_I = \frac{P_2(x_1,x_2)}{Q_2(x_1,x_2)} = \frac{c_1x_1 + c_2x_2 + c_0}{d_1x_1 + d_2x_2 + d_0} \to \text{opt}(\max,\min), \tag{13}$$

Let us first consider a case when the objective function of the problem is homogeneous (12). We know that the solution to system of constraints $\Omega_I$ of problem (11) is a convex set constrained by lines also called polyhedron. In general case, $\Omega_I$ is geometrically shown as a polygon (Fig. 1).

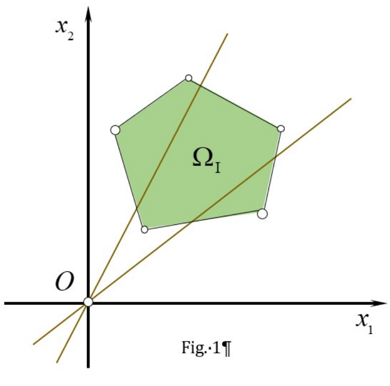


**Figure 1:** Case, $\Omega_I$

For geometric interpretation of conduct of objective function (12), solve this equation relative to $x_2$:

$$W_I \, d_1x_1 + W_I \, d_2x_2 = c_1x_1 + c_2x_2,$$

$$\left(W_I \, d_2 - c_2\right)x_2 = \left(c_1 - W_I \, d_1\right)x_1,$$

$$x_2 = \frac{c_1 - W_I d_1}{W_I d_2 - c_2} x_1.$$

We introduce designation

$$k_{12} = \frac{c_1 - W_I d_1}{W_I d_2 - c_2},$$

and then we obtain the equation of line $\omega$

$$\omega: x_2 = k_{12} x_1,$$

that passes across the coordinate's origin $O$.

Providing various values to objective function $W_I$, we obtain a sheaf of lines with the center at point $O$ of the coordinate's origin. The sense of geometric solution of the two-dimensional linear fractional optimization problem consists in finding such a line that corresponds to the optimum value of $W_I$ and belongs to polyhedron $\Omega_I$ at the same time. This line that is commonly called tagline (Fig. 2) will be a line touching the apex $\Omega_I$ or passing across the polyhedron side corresponding to the alternative minimum case (Fig. 3).

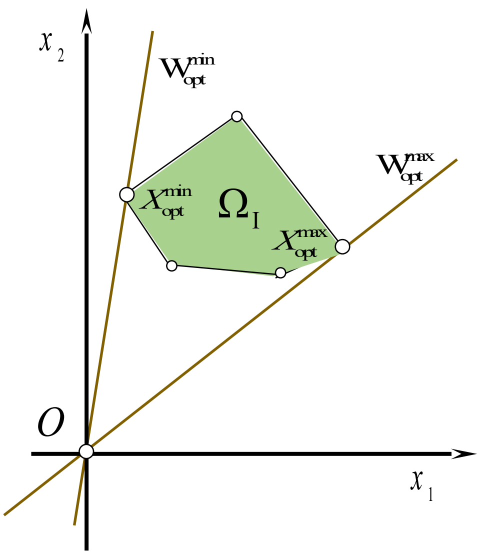


**Figure 2:** A tagline.

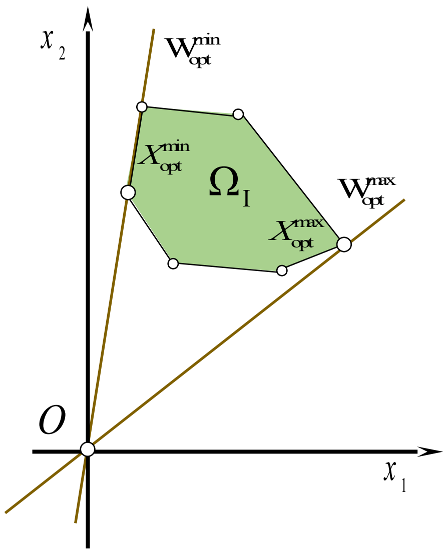


**Figure 3:** A line touching the vertex or passing through the side of the polyhedron corresponding to the alternative minimal case.

The coordinates of the apexes the tagline passes across are the very coordinates giving the optimum problem attack plans.

Depending on the two-dimensional system of constraints $\Omega_I$, the following cases are possible: (Fig. 4)

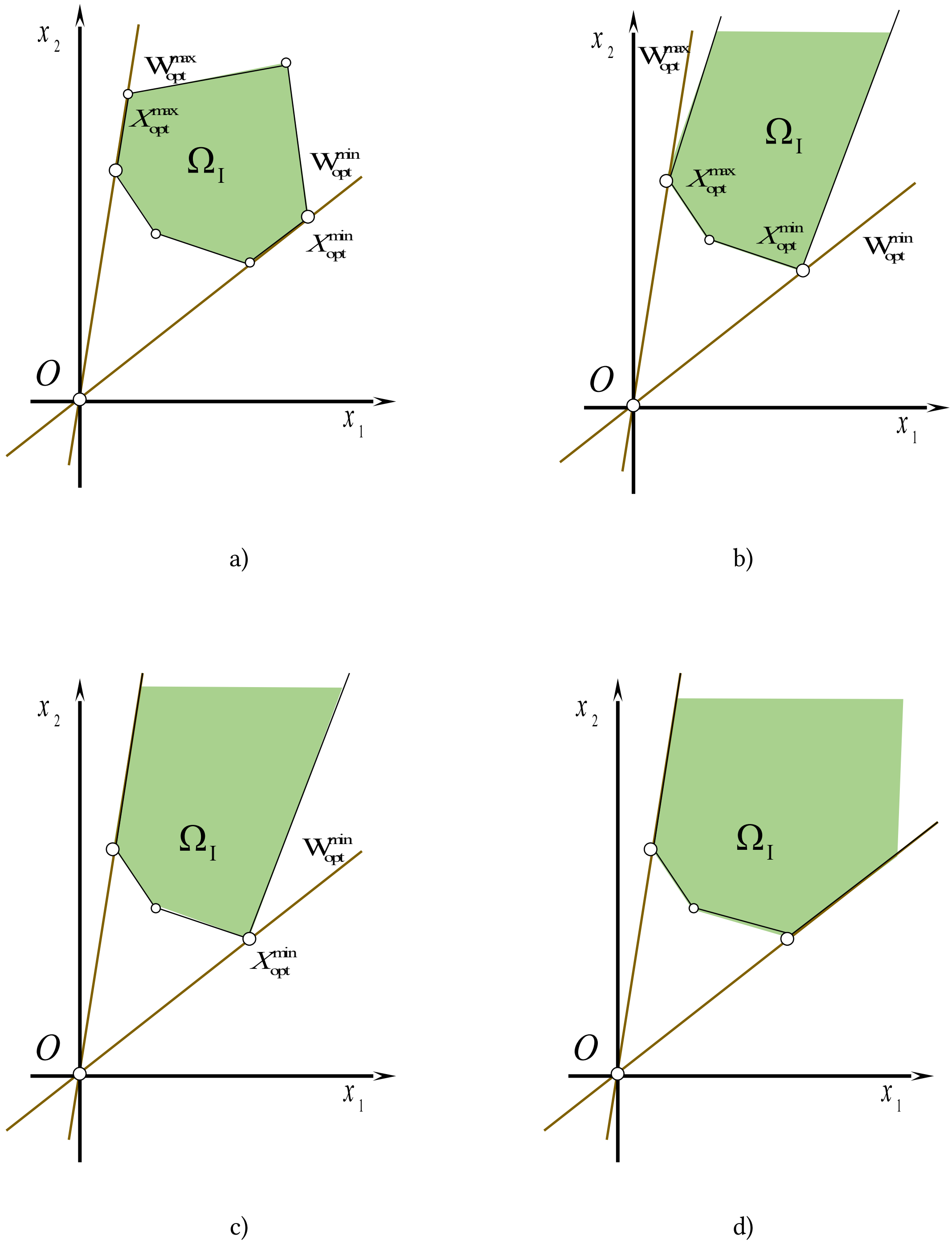


**Figure 4:** Various options for the location of the tagline depending on the two-dimensional constraint system.

- The admissible values area $\Omega_I$ is constrained, there are two alternative optimums obtained at points of the two sides of polyhedron $\Omega_I$ (Fig. 4a)
- The admissible values area $\Omega_I$ is not constrained, there are two angular apexes providing the optimum values to the objective function (Fig. 4b)
- The admissible values area $\Omega_I$ is not constrained, there is only one angular apex providing the optimum (the minimum) value to the objective function. The second optimum (minimum) corresponds to a case of the asymptotic maximum (Fig. 4c)

- The admissible values area $\Omega_I$ is not constrained. The optimums are asymptotic (Fig. 4d)

Let us consider some examples of a geometric solution to linear fractional optimization problems in case of a homogeneous objective function.

## 4. Conclusion

Linear fractional optimization is a powerful tool in project management, particularly when optimizing complex relationships between cost, time, and quality. By employing this method, project managers can achieve more balanced and practical solutions that ensure an efficient allocation of resources while simultaneously minimizing risks. This approach is beneficial in scenarios where traditional optimization techniques may need to fully address the multidimensional nature of project constraints and trade-offs.

One of the critical strengths of linear fractional optimization is its ability to handle conflicting objectives, such as reducing expenses while maintaining high quality and meeting tight deadlines. These trade-offs are common in project management, where stakeholders often have differing priorities and limited resources. The method provides a structured way to identify the best possible outcomes, making it easier to align the project goals with available resources and strategic objectives.

Despite the inherent complexity of many projects, linear fractional optimization remains adaptable and flexible. It offers a clear framework for decision-making in both simple and highly intricate project environments, making it a versatile tool. Its adaptability to various industries, from construction and engineering to information technology and healthcare, where optimizing performance, cost efficiency, and quality assurance are critical, further underscores its adaptability and potential impact.

As project management continues to evolve with new technologies and methodologies, the practicality and relevance of linear fractional optimization become more apparent. The potential for further development and broader application of this technique grows, especially with advances in computational tools, data analytics, and artificial intelligence. These advancements make linear fractional optimization more accessible and capable of handling even more complex decision-making processes, thereby improving project outcomes.

## Declaration on Generative AI

The authors have not employed any Generative AI tools.

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