Duffing pendulum model in the non-orthogonal coordinates

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Abstract

Our paper deals with the design of mathematical backgrounds to study 2nd-order dynamical systems' phase portraits in planes with non-orthogonal coordinates. We design our approach by using affine coordinate transformations such as shift, rotation, and scale to define the new system phase portrait. Since coordinate transformations are defined by algebraic expressions one can find it is not convenient to use both differential equations to define system motions and algebraic ones to perform the transformations. That is why we offer to differentiate transformation expressions and use the obtained derivatives to write down system differential equations in the transformed coordinate system. Such differentiation in the most common case gives us the possibility to take into account coordinate systems motions. Moreover, one can define a coordinate system in which motions depend on the motions of the initial dynamical system and time. So, in the most general case, our transformation allows us to define the variable-structure time-dependent systems. We show the use of our approach by considering the dynamic of the Duffing pendulum in the non-orthogonal coordinates. Such transformation makes it possible to design and implement a chaotic generator that forms oscillations different from known ones.

Keywords

dynamical system, coordinate transformation, chaotic system, Duffing equation

1. Introduction

Today various nonlinear systems are widely used in different scientific and engineering applications because their unique properties [1, 2, 3]. These systems are used to study various biological, meteorological, communication and process control, and technical systems modeling [4, 5, 6, 7, 8, 9, 10].

One of the applications for such systems is control plant prototyping to design novel control algorithms which are used in cyber-physical systems [11, 12, 13, 14, 15] and Industrial Internet of Things' systems [16, 17, 18, 19, 20, 21].

A lot of various systems with nonlinear dynamic are known [22, 23, 24, 25, 26, 27, 28] and used [29, 30, 22, 23] in different communication applications as well [31, 32, 33]. However, many authors design and study these systems without any explanation of how and why they use one or another block in their system. Thus, the main drawback of known chaotic systems is their subjective design.

We offer to avoid this drawback by using a well-known coordinate transformation to give the considered system the desired features and put it attractor in the desired domain of coordinate system. Our method is demonstrating by considering a simple 2nd order chaotic system in the 2D plane, but due to the matrix methods usage it can easily be extended to any N-dimensional dynamical system.

The paper is organized as follows: firstly, we affine transformation from one orthogonal coordinate system into another one and define system matrix motion equation as derivative of transformation equation. Then, we generalize this equation for the case of non-orthogonal coordinates. We illustrate

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the use of our approach by studying a well-known Duffing equation's transformation and implementing it by using modern single-chip MCU. Finally, we make a conclusion.

2. Method

2.1. Dynamical system model in the orthogonal coordinate system

Let us show the principle of the proposed dynamical system transformation by considering the 2nd order harmonic-driven nonlinear dynamical system

$$\ddot{y}_1 = f(y_1, \dot{y}_1) + \gamma \sin(\omega t + \varphi_0), \tag{1}$$

where y_1 is a system state variable, $f(.)$ is some nonlinear function, γ and ω are parameters of an external harmonic signal.

But our studies can be easy generalized for the case of N-dimensional dynamical system as well.

If one takes into the consideration derivative of system state output as some new state variable , he can rewrite (1) in the normal canonical form

$$\dot{y}_1 = y_2; \; \dot{y}_2 = f(y_1, y_2) + \gamma \sin(\omega t + \varphi_0) \tag{2}$$

and then represent it as a matrix equation

$$\begin{aligned} &\dot{\boldsymbol{Y}} = \boldsymbol{F}(\boldsymbol{Y}) + \boldsymbol{G}(\omega, t, \varphi_0), \\ &\boldsymbol{Y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T; \; \boldsymbol{F}(\boldsymbol{Y}) = \begin{pmatrix} y_2 f(y_1, y_2) \end{pmatrix}; \; \boldsymbol{G}(\omega, t, \varphi_0) = \begin{pmatrix} 0 \\ \gamma \sin(\omega t + \varphi_0) \end{pmatrix}. \end{aligned} \tag{3}$$

In the most general case, here and further one should use n-th sized vectors and matrices to define system dynamic in N-dimensional state space and perform its transformations. Solution of this system allows us to define some phase portrait $y_1(y_2)$ (Figure. 1).

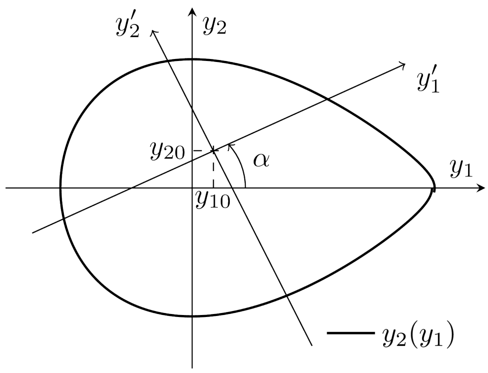

Figure 1: Phase portrait of nonlinear dynamical system in the orthogonal axes

If one applies affine transformations to the system phase portrait, he can define the system motions in another coordinate system $y_1'(y_2')$. We think that old $y_1(y_2)$ and new $y_1'(y_2')$ coordinate systems are interrelated each other by using gain, shift, and rotate transformations, which allows us to define system's state variables in the new coordinates by using following matrix expression

$$\begin{aligned} &\boldsymbol{Y'} = \boldsymbol{KRY} + \boldsymbol{Y_0}, \\ &\boldsymbol{Y'} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}; \; \boldsymbol{K} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}; \; \boldsymbol{Y_0} = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}; \; \boldsymbol{R} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \end{aligned} \tag{4}$$

here k_i are some gain factors, y_{0i} are shift factors, and α is angle between new and old coordinate systems (Figure 1), $\boldsymbol{K}$ is a gain matrix, $\boldsymbol{R}$ is a rotation matrix, and $\boldsymbol{Y_0}$ is a shift matrix. Here and later we assume that all terms in matrices $\boldsymbol{K}$, $\boldsymbol{Y_0}$, and R are time-depended ones.

From the control theory viewpoint, one can consider (4) as observability equation for systems (3) and so use control methods to study systems that are defined by the equations.

Solution of (3) and substituting its results into (4) gives us the possibility to define system coordinates in the new coordinate system that is different from the initial one. The main drawback of such an approach is representing system motion by both algebraic and differential equations which can cause some difficulties in applying the known control theory approaches and methods to study system stability or design its motion trajectories.

We offer to avoid this drawback by transforming (3) and (4) into differential equation only. To perform such a transformation, we take into account the-above made assumption about matrices terms and differentiate (4)

$$\dot{\boldsymbol{Y'}} = \dot{\boldsymbol{Y_0}} + \dot{\boldsymbol{K}}\boldsymbol{R}\boldsymbol{Y} + \boldsymbol{K}\dot{\boldsymbol{R}}\boldsymbol{Y} + \boldsymbol{K}\boldsymbol{R}\dot{\boldsymbol{Y}}. \tag{5}$$

Since the last term in (5) depend on derivatives of system state vector we substitute (3) into (5)

$$\dot{\boldsymbol{Y'}} = \dot{\boldsymbol{Y_0}} + \dot{\boldsymbol{K}}\boldsymbol{R}\boldsymbol{Y} + \boldsymbol{K}\dot{\boldsymbol{R}}\boldsymbol{Y} + \boldsymbol{K}\boldsymbol{R}(\boldsymbol{F}(\boldsymbol{Y}) + \boldsymbol{G}(\omega, t, \varphi_0)). \tag{6}$$

It is clear that right-hand expression in (6) depends on system motion vector $\boldsymbol{Y}$ in the old coordinate system. This fact causes some misunderstanding in solution of (6) which we avoid by solving (4) for $\boldsymbol{Y}$.

$$\boldsymbol{Y} = inv(\boldsymbol{K}\boldsymbol{R})(\boldsymbol{Y'} - \boldsymbol{Y_0}), \tag{7}$$

where $inv(.)$ means determination of inverse matrix,

and substituting (7) into (6)

$$\dot{\boldsymbol{Y'}} = \dot{\boldsymbol{Y_0}} + (\dot{\boldsymbol{K}}\boldsymbol{R} + \boldsymbol{K}\dot{\boldsymbol{R}})inv(\boldsymbol{K}\boldsymbol{R})(\boldsymbol{Y'} - \boldsymbol{Y_0}) + \boldsymbol{K}\boldsymbol{R}(\boldsymbol{F}(inv(\boldsymbol{K}\boldsymbol{R})(\boldsymbol{Y'} - \boldsymbol{Y_0})) + \boldsymbol{G}(\omega, t, \varphi_0)). \tag{8}$$

We call (8) as dynamical system's motion equation in the generalized orthogonal coordinate system. The main feature of this equation is possibility to take into account motion of coordinate system's origin, axes rotation and scaling as some known time functions.

The particular case of this equation is a system motion's equation in the stationary rotated, shifted, and scaled coordinate system, which allows us to neglect the first summand in (8)

$$\dot{\boldsymbol{Y'}} = \boldsymbol{K}\boldsymbol{R}(\boldsymbol{F}(inv(\boldsymbol{K}\boldsymbol{R})(\boldsymbol{Y'} - \boldsymbol{Y_0})) + \boldsymbol{G}(\omega, t, \varphi_0)). \tag{9}$$

One can use (9) and (8) to define novel motions for known dynamical systems. It is clear that for the nonstationary coordinate systems these motions can significantly differ from known ones.

2.2. Dynamical system model in the non-orthogonal coordinate system

One of the features of (9) and (8) is a rotation matrix $\boldsymbol{R}$. This matrix has the simplest form in the case of orthogonal coordinate system. At the same time one can define transformation (4) for the case non-orthogonal coordinates as well. The necessity to consider such coordinate system can occurs in the case when angle between axes is different than $\pi/2$, it corresponds the use of the generalized differential operators which forms phase shift different from $\pi/2$. For example, in fractional order systems or other which use both classical derivatives from system state variables and their current values.

Let us consider the transformation of such a system in details we think that the angle between axes is equal to some angle β. Performing of simple trigonometric transformations allows us to rewrite transformation matrix $\boldsymbol{R}$ as follows

$$\boldsymbol{R} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \cos(\alpha + \beta) & \sin(\alpha + \beta) \end{pmatrix}. \tag{10}$$

It is clear that matrix $\boldsymbol{R'}$ in (10) is the generalization of matrix $\boldsymbol{R}$ in (4) and for $\beta = 0$ it allows to define only one axis . The position of dynamical system in this axis depend on α in this case. When β is considered as $\pi/2$ the previously defined $\boldsymbol{R}$ matrix can be obtained.

Replacing $\boldsymbol{R}$ with $\boldsymbol{R'}$ in (8) gives us the possibility to define the generalized system motion equation for the case of non-orthogonal coordinate system

$$\begin{aligned}\dot{\boldsymbol{Y'}} &= \dot{\boldsymbol{Y_0}} + (\dot{\boldsymbol{K}}\boldsymbol{R'} + \boldsymbol{K}\dot{\boldsymbol{R'}})inv(\boldsymbol{KR})(\boldsymbol{Y'} - \boldsymbol{Y_0}) + \\ &+ \boldsymbol{KR'}(\boldsymbol{F}(inv(\boldsymbol{KR'})(\boldsymbol{Y'} - \boldsymbol{Y_0})) + \boldsymbol{G}(\omega, t, \varphi_0)).\end{aligned} \tag{11}$$

Thus, in the most common case, (11) can be used to define motion equations of the generalized dynamical system in some arbitrary coordinate system which can be non-orthogonal one.

3. Results and discussions

We show the use of our approach by studying a well-known Duffing's pendulum equation

$$\ddot{y} + a_1\dot{y} + a_2y + a_3y^3 = b_1\cos(\omega t), \tag{12}$$

here $\ddot{y}$, $\dot{y}$, and y are pendulum acceleration, speed, and position, a_i are pendulum parameters, b_1 and ω are driving force factors.

Since our approach is based on the use of matrix equations let us rewrite (12) into matrix form (3) with matrices

$$\boldsymbol{Y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T; \ \boldsymbol{F(Y)} = \begin{pmatrix} y_2 - a_1y_2 + a_2y_1 - a_3y_1^3 \end{pmatrix}; \ \boldsymbol{G}(\omega, t, \varphi_0) = \begin{pmatrix} 0 \\ b_1\cos(\omega t) \end{pmatrix}.$$

At first, we consider the stationary case of coordinate system transformation from the orthogonal into some nonorthogonal (9) as more simple case and then generalize out results for the case of model transformation into non-stationary coordinates.

Let assume that this transformation is characterized by matrices (4) and (10), and rewrite (7) as follows

$$\boldsymbol{Y} = inv\left(\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \cos(\alpha+\beta) & \sin(\alpha+\beta) \end{pmatrix}\right)\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} - \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \tag{13}$$

$$= \begin{pmatrix} \frac{\sin(\alpha+\beta)}{k_1\sin(\alpha)}(y_1' - y_{10}) - \frac{\sin(\alpha)}{k_2\sin(\beta)}(y_2' - y_{20}) \\ -\frac{\cos(\alpha+\beta)}{k_1\sin(\alpha)}(y_1' - y_{10}) + \frac{\cos(\alpha)}{k_2\sin(\beta)}(y_2' - y_{20}) \end{pmatrix}. \tag{14}$$

Then, we substitute (13) into matrix-function $\boldsymbol{F(Y)}$

$$\boldsymbol{F(Y)} = \begin{pmatrix} k_{11}(y_1' - y_{10}) + k_{12}(y_2' - y_{20}) \\ k_{22}(y_2' - y_{20}) + k_{21}(y_1' - y_{10}) + a_3(k_{232}(y_2' - y_{20}) + k_{231}(y_1' - y_{10}))^3 \end{pmatrix}, \tag{15}$$

$$k_{231} = \frac{\cos(\alpha)}{k_2\sin(\beta)}; \ k_{11} = \frac{\sin(\alpha+\beta)}{k_1\sin(\alpha)}; \ k_{22} = \frac{a_1\cos(\alpha) - a_2\sin(\alpha)}{k_2\sin(\beta)}; \ k_{232} = -\frac{\cos(\alpha+\beta)}{k_1\sin(\alpha)};$$
$$k_{12} = -\frac{\sin(\alpha)}{k_2\sin(\beta)}; \ k_{21} = \frac{a_2\sin(\alpha+\beta) - a_1\cos(\alpha+\beta)}{k_1\sin(\beta)}.$$

Analysis of (15) shows that contrary to (13) the terms of vector-function (15) depends on both pendulum parameters and transformation factors. Moreover, detailed study of the first row in matrix (15) shows that after transformation the pendulum position depends on both its speed and position. This fact proves the above-given sentence about the generalized differential operator because control theory shows that phase shift in dynamical system with inner feedback is not equal $\pi/2$.

If one substitutes (15) and (13) into (9) he can write down the Duffing equation in the non-orthogonal coordinate system

$$\dot{\boldsymbol{Y'}} = \begin{pmatrix} d_{110}y_{10} + d_{11}y_1' + d_{120}y_{20} + d_{12}y_2' + \\ +d_{13}(k_{232}(y_{20} - y_2') + k_{231}(y_{10} - y_1'))^3 + e_1\cos(\omega t) \\ d_{210}y_{10} + d_{21}y_1' + d_{220}y_{20} + d_{22}y_2' + \\ +d_{23}(k_{232}(y_{20} - y_2') + k_{231}(y_{10} - y_1'))^3 + e_2\cos(\omega t) \end{pmatrix}, \tag{16}$$

$$d_{110} = k_1 k_{21} \sin(\alpha) - k_1 k_{11} \cos(\alpha);\ d_{11} = k_1 k_{11} \cos(\alpha) - k_1 k_{21} \sin(\alpha);$$
$$d_{120} = -k_1 k_{12} \cos(\alpha) + k_1 k_{22} \sin(\alpha) l\ d_{12} = k_1 k_{12} \cos(\alpha) - k_1 k_{22} \sin(\alpha);$$
$$e_1 = k_1 b_1 \sin(\alpha);\ d_{210} = k_2 k_{21} sin(\alpha + \beta) - k_2 k_{11} \cos(\alpha + \beta);$$
$$d_{22} = -k_2 k_{22} \sin(\alpha + \beta) + k_2 k_{12} \cos(\alpha + \beta);$$
$$e_2 = k_2 b_1 \sin(\alpha + \beta);\ d_{23} = k_2 a_3 \sin(\alpha + \beta).$$

Analysis of (16) shows that the equation has the similar components but with different factors. We believe that this fact can be used while control problems are being solved to reduce the order of controlled system but checking of this hypothesis is leaved for future studying.

Numerical solution of (16) is performed by using a well-known Arduino Due board. Numerical simulation program, which solves (16), implements the simplest integration method which is based on the use of finite backward difference with sample time 0.001 s. Signals from the board where obtained to PC by using standard serial communication. In Figure.2 and Figure.3 we show the transformed and classical system attractors.

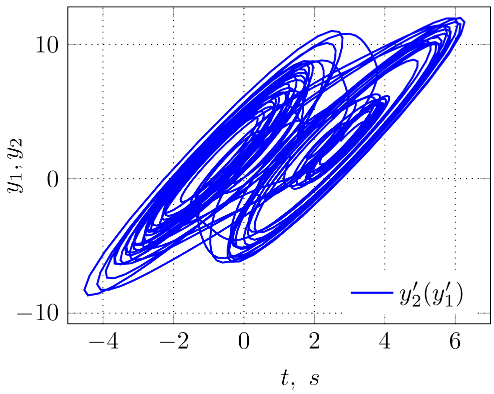

Figure 2: Transformed pendulum attractor

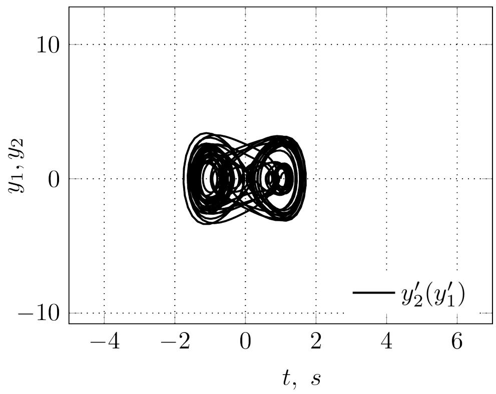

Figure 3: Classical pendulum attractor

As one can see our approach allows to scale, rotate, and shift the transformed attractor in comparison with the known one. The above-given results are obtained for the following transformation factors here , $\alpha = pi/6$, $\beta = pi/4$ and pendulum equation factors $a_1 = 1$, $a_2 = 0.02$, $a_3 = 5$, $b_1 = 8$, $\omega = 0.5$.

It is clear that the attractor changing is caused by changing chaotic oscillations in the transformed dynamical system. We show transformed and classical pendulums oscillations in Figure.4 and Figure.5 respectively.

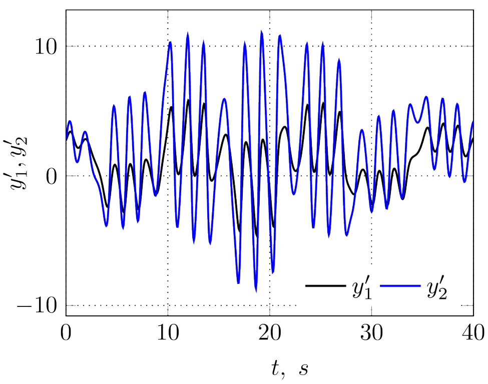

Figure 4: Numerical solution of the transformed Duffing equation

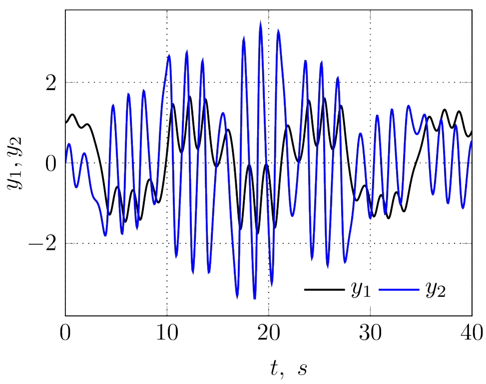

Figure 5: Numerical solution of the classical Duffing equation

Analysis of the above-given results of numerical simulations allows us to claim that even in a quite short time range statically-transformed and classical systems produce different outputs. This fact allows us to claim the possibility to generate novel chaotic oscillations by transforming known ones. Here we study the simplest affine transformation with the constant factors but we believe that the use of nonlinear one with variable factors which depend on system state variables and time can dramatically change system motion.

Now we turn our attention into the most general case when transformations factors are some time-depended functions. In this case we use (11) instead of (9) to perform the pendulum model transformation. Also, we take into account that the last summand in (11) is the system motion equation in stationary coordinates which is defined by (9). This fact allows us to use the previously defined equation (16) as the part of following one

$$\dot{\boldsymbol{Y'}} = \left(\begin{array}{c} k_2(y_1 - y_{01})\dot{k}_1 \sin(\alpha) + k_1\dot{\alpha}\cos(\alpha)\cos(\alpha+\beta)+ \\ +k_2(y_2 - y_{02})\dot{k}_1 \sin(\alpha) + k_1\dot{\alpha}\cos(\alpha)\sin(\alpha+\beta)+ \\ +k_1^2\dot{\alpha}((y_2 - y_02)\cos^2(\alpha) - sin(\alpha)(y_1 - y_{01})\cos(\alpha) - y_2 + y_{02})\dot{\alpha}+ \\ +k_1((y_1 - y_{01})\cos(\alpha) + \sin(\alpha)(y_2 - y_{02}))\cos(\alpha)\dot{k}_1 + \dot{y}_{01}+ \\ +d_{110}y_{10} + d_{11}y_1' + d_{120}y_{20} + d_{12}y_2'+ \\ +d_{13}(k_{232}(y_{20} - y_2') + k_{231}(y_{10-y_1'}))^3 + e_1\cos(\omega t) \\ -(\dot{k}_2(y_2 - y_{02}) - k_2(y_1 - y_{01})(\dot{\alpha} + \dot{\beta}))k_2\cos^2(\alpha+\beta)+ \\ +(k_2((y_1 - y_{01})(\dot{k}_2 + k_2(y_2 - y_02)(\dot{\alpha} + \dot{\beta}))\sin(\alpha+\beta)+ \\ +k_1\dot{k}2((y_1 - y_{01})\cos(\alpha) + \sin(\alpha)(y_2 - y_{02}))\cos(\alpha+\beta)- \\ -(\dot{\alpha} + \dot{\beta})k_1((y_1 - y_{01})\cos(\alpha) + \sin(\alpha)(y_2 - y_{02}))k_2\sin(\alpha+\beta)+ \\ +k_2(y_2 - y_{02})\dot{k}_2 + \dot{y}_{02} + d_{210}y_{10} + d_{21}y_1' + d_{220}y_{20} + d_{22}y_2'+ \\ +d_{23}(k_{232}(y_{20-y_2'}) + k_{231(y_{10}-y_1')})^3 + e_2\cos(\omega t) \end{array} \right) \tag{17}$$

Equation (17) use the most general terms to define derivatives for components of matrices $\boldsymbol{K}$, $\boldsymbol{Y0}$, and $\boldsymbol{R'}$ by assuming all these components are some functions. The main feature of the transformed system is its determination by using both polynomial and trigonometric functions. One can use this fact to define the classes of system's nonlinearities which use cause chaotic motions.

One can specify (17) by taking into account different transformation functions' formulas. It is clear that there is no any restriction of the functions' formulas. Also, we think that these functions make great effect in system dynamic that is why we leave study and design of chaotic systems with specified transformation functions for future research and now show some examples of numerical solution (17). We use the same numerical methods to solve this equation as the above-used. Also, we leave unchanged the MCU board to implement the considered chaotic system.

Implementation of all above-considered dynamical systems are performed by using the simplest feedback difference approximation for derivative operator

$$\dot{y} \approx \frac{1 - z^{-1}}{h} y, \tag{18}$$

here h is a system discretization period and z^{-1} means backward signal shifting on one discretization period.

It is clear that contrary to transformation of pendulum motion for the case of constant values of transformation factors, in the most general case of variable transformation factors one should take into account both value and derivative of such factor. This fact causes the necessity to define trajectories of these factors' motions.

Since these trajectories can be defined by tons of different expressions, we consider here several of them and leave detailed system studies for future research.

We start our studies from the case of linear time-depended changing of transformation factors

$$w_i = w_{i1}t + w_{i0}, \tag{19}$$

where w_i is a generalized transformation factor, w_{ij} are parameters of linear dependency and t is a system time.

Simulation results for the system (17) with transformation factors (19) are shown if Figure.6-Figure.8

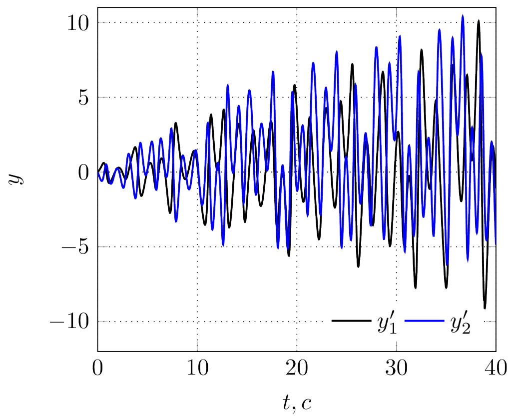

Figure 6: Pendulum transformed state variables in non-orthogonal linearly-moved coordinates

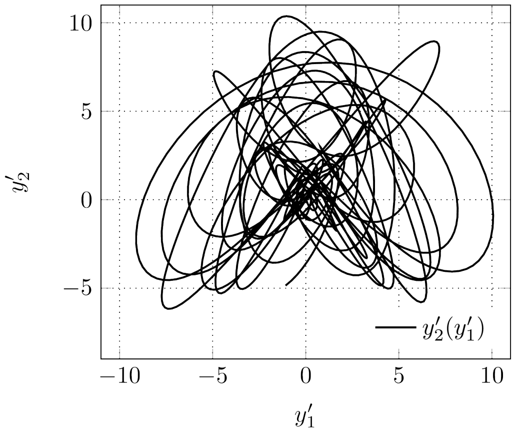

Figure 7: Transformed pendulum attractor in non-orthogonal linearly-moved coordinates

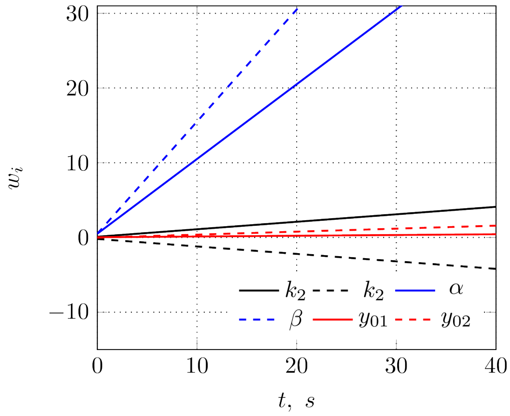

Figure 8: Pendulum factors

Shown in Figure.6-Figure.8 simulation results are obtained for following linear time-depended transformation factors

$$\begin{aligned} &k_1 = 0.1t + 0.1;\ k_2 = -0.1t - 0.2;\ y_{01} = 0.01t + 0.03; \\ &y_{02} = 0.04t - 0.02;\ \alpha = t + 0.5;\ \beta = 1.5t + 0.5. \end{aligned} \tag{20}$$

Analysis of given in Figure.6-Figure.7 simulation results shows that system trajectories after transformation into moved non-orthogonal coordinates dramatically changed in comparison of the initial system. This fact makes strong backgrounds to the design of novel systems. The motion equations for these systems can be defined in terms of system output variable and its derivatives as well as the system time. Thus, the usage of transformation factors (20) allows us to define the new class of systems with chaotic dynamic which motions depend on both state variables and time.

It is clear that the linear dependency of transformation factors from system time define linear rising of system state variables because of the continuous linear origin shifting and gains increasing.

For the case of polynomial dependency of transformation factors from time

$$w_i = w_{i1}t^2 + w_{i0}, \tag{21}$$

one can obtain following simulation results (Figure.9-Figure.11),

Shown in Figure.9-Figure.11 simulation results are obtained for following polynomial time-depended transformation factors

$$\begin{aligned} &k_1 = 0.0025(t+1)^2;\ k_2 = -0.0025(t+2)^2; \\ &y_{01} = 0.00025(t+3)^2;\ y_{02} = 0.00025(4t-2)^2; \end{aligned} \tag{22}$$

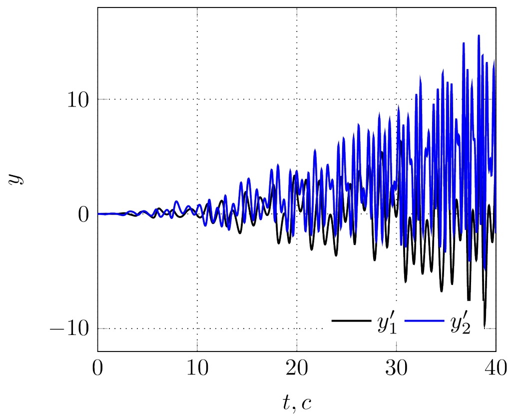

Figure 9: Pendulum transformed state variables in non-orthogonal polynomialy-moved coordinates

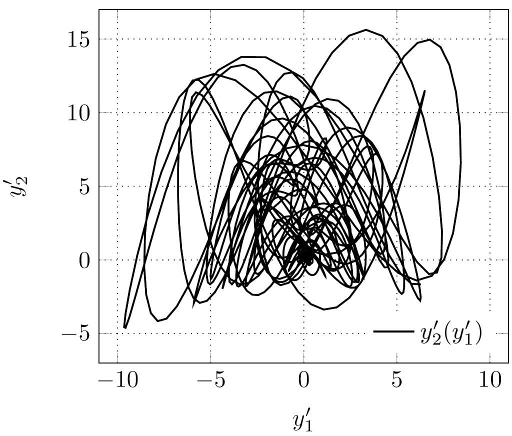

Figure 10: Transformed pendulum attractor in non-orthogonal polynomialy-moved coordinates

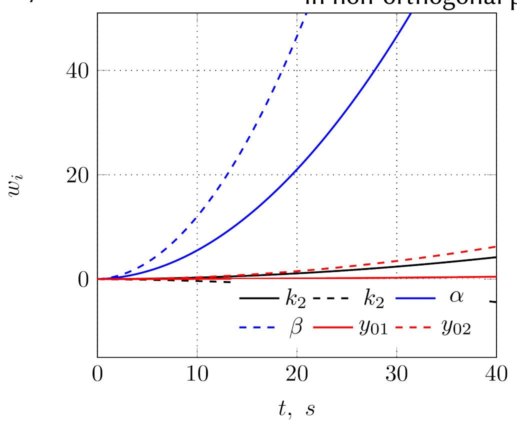

Figure 11: Pendulum factors

$$\alpha = 0.0125(2t+1)^2;\ \beta = 0.0125(3t+1)^2.$$

Analysis of given inFigure.9-Figure.11 simulation results shows that the use of nonlinear time-depended functions to define transformation factors makes system dynamic more complex and allows to increase the oscillation frequency. Also, it allows us to claim that the amplitude of chaotic oscillations in the transformed coordinates depends from the functions which define the transformation factors. This sentence is proven by quadratic rising of oscillation amplitudes which depend on system time.

The above-given simulation results allow us to claim that use of more complex functions to define the transformation factors allows us to form highly-nonlinear system motions. To prove this sentence, we consider following harmonic transformation functions

$$k_1 = \sin(t+1);\ k_2 = \cos(t+2);\ y_{01} = \sin(0.1t+0.3);\ y_{02} = \cos(0.4t-0.2); \tag{23}$$

$$\alpha = \sin(t+0.5);\ \beta = \cos(1.5t+0.5). \tag{24}$$

Simulation results for dynamical system(17) which use (23) are shown in Figure.12-Figure.14.

Analysis of the given in Figure.7 curves shows that one can use highly-nonlinear time-depended functions to perform system transformation and design novel chaotic system which are defined by nonlinear equations.

All above-considered transformed systems are designed by using information about system time to define the transformation functions. It is clear that such approach is convenient from practical implementation viewpoint only in case when bounded or harmonic function are used. Because in case of unbounded functions after achieving some time value system coordinates can reach very high values which cannot be physically implemented. So, one should use time-depended transformation factors carefully. At the same time the using information about system state variables allows us to solve this problem and design novel chaotic system without using system time in explicit form.

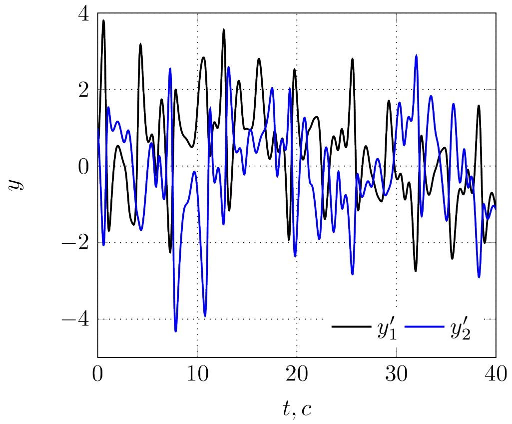

Figure 12: Pendulum transformed state variables in non-orthogonal harmonicaly-moved coordinates

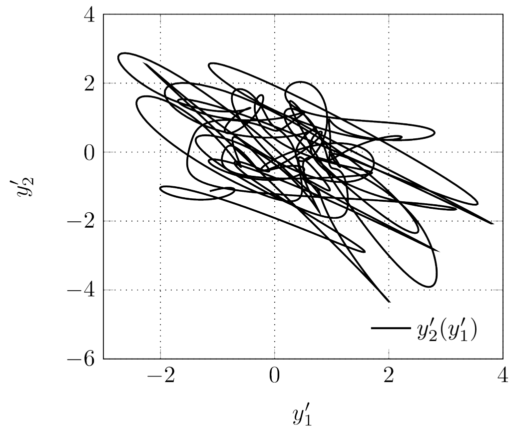

Figure 13: Transformed pendulum attractor in non-orthogonal haarmonicaly-moved coordinates

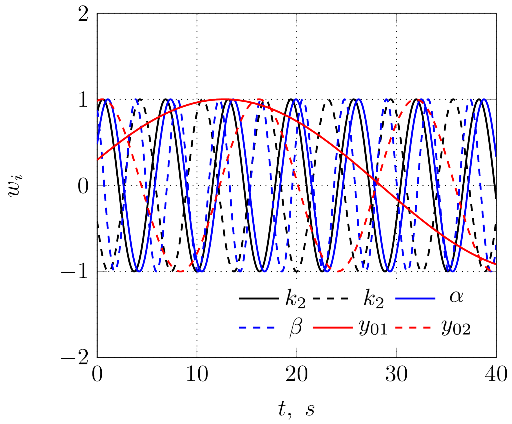

Figure 14: Pendulum factors

Thus, in Figure.8 and Figure.9 we show the simulation results for the case when time in (20) and (23) are replaced with pendulum state variables. This replacement is defined by using cross-numbering strategy, according to which the factor w_1 depends on y_2 and vice versa.

The given simulation results prove above-given hypothesis about possibility to use the pendulum state variables to define transformation factors and allow us to claim that highly-nonlinear chaotic system can be designed in case of using nonlinear transformation functions which depend on system state variables.

4. Conclusions

Applying of known affine transformations to the phase portrait of the generalized dynamical system allows us to transform this phase portrait into non-orthogonal coordinate system. From the control theory viewpoint, in this case, the motion of transformed system is defined with system of matrix differential motion equation and algebraic observability equation. To fit the use of these equations to known methods of dynamical system analysis and synthesis we represent them with only differential equations by differentiating the observability equation. Such an approach allows us to take into account possible motions of the new coordinate system, where the motion of considered dynamical system are transformed, and define matrix expression for this transformation. The use of this expression allows us to write down differential equations for any transformed dynamical system without performing any coordinate transformation and using observability equation.

Applying of the proposed approach to the existing chaotic systems equations allows us to define novel chaotic systems which attractors are defined in non-orthogonal coordinate systems. Our studies

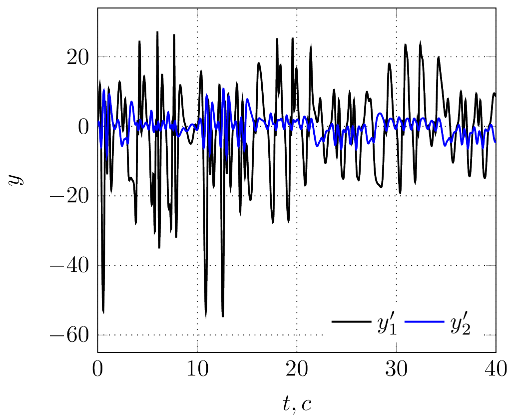

Figure 15: Pendulum transformed state variables in non-orthogonal linearly-moved coordinates which linearly depend on pendulum state variables

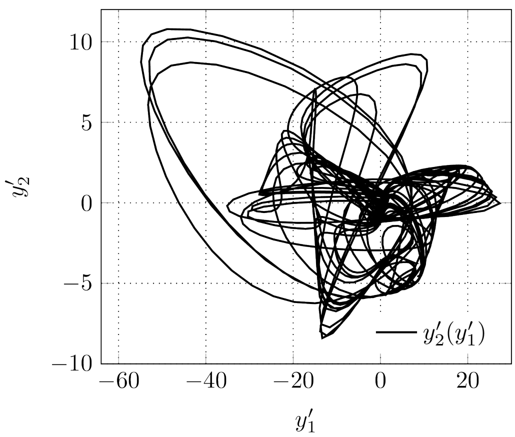

Figure 16: Transformed pendulum attractor in non-orthogonal linearly-moved coordinates which linearly depend on pendulum state variables

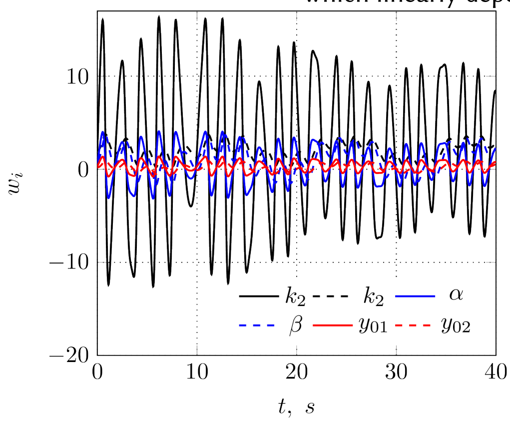

Figure 17: Pendulum factors

show that using of the above-given approach while Duffing pendulum is being modeled, allows to change the form of its oscillations and shift, scale and rotate its attractor.

We see the future development of our work in applying nonlinear transformations to the considered nonlinear system.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

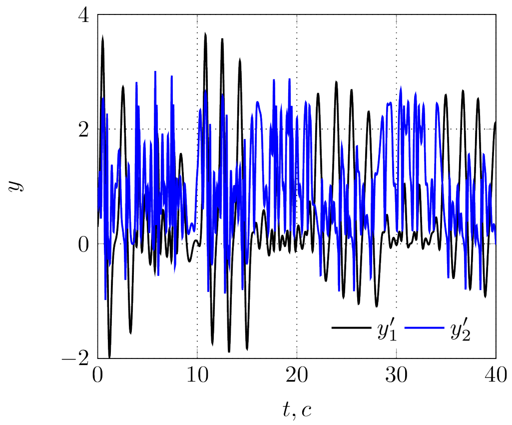

Figure 18: Pendulum transformed state variables in non-orthogonal harmonicaly-moved coordinates which harmonicaly depend on pendulum state variables

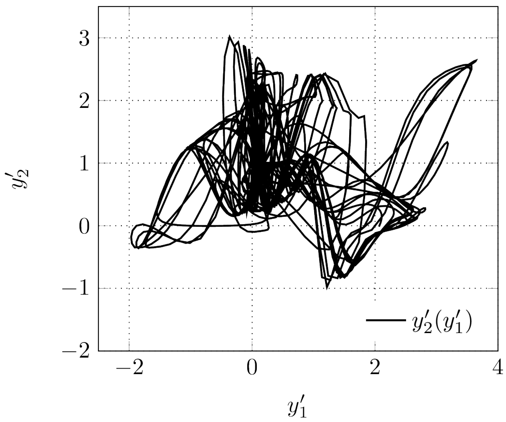

Figure 19: Transformed pendulum attractor in non-orthogonal harmonicaly-moved coordinates which harmonicaly depend on pendulum state variables

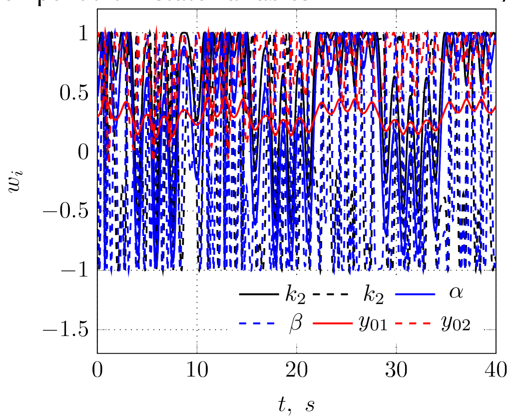

Figure 20: Pendulum factors

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