Perturbed motion of two coupled Duffing pendulums

Roman Voliansky[1,*,†], Nina Volianska[2,†] and Aji Prasetya Wibawa[3,†]

[1]National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Beresteiskyi Ave., 37, Kyiv, 03056, Ukraine
[2]Taras Shevchenko National University of Kyiv, Bohdan Havrylyshyn Str., 24 , Kyiv, 04116, Ukraine
[3]State University of Malang, Jl. Cakrawala No.5, Sumbersari, Kec. Lowokwaru, Kota Malang, Jawa Timur, 65145, Indonesia

Abstract

The paper deals with the design of backgrounds to study and construct coupled single and multichannel dynamical systems. Our studies are based on considering the motion equations of the known dynamical systems and defining interrelations between these systems. Such an approach allows the transformation of one class of time-variant systems into a time-invariant one, and motion analysis for them can be performed using the known control theory methods. We study the motions of each subsystem and consider their trajectory variations to define the system's perturbed motions. The motions' equations can be determined by taking into account the system model and differentiating the perturbed motion coordinates. Such an approach allows us to define system dynamics as a function of the perturbed motion coordinates and their derivatives only and does not require solving equations of the initial system. The coupled system perturbed motion differs from the initial ones and allows us to consider the perturbed motion's dynamical systems as novel systems. Our method is proven by considering the well-known Duffing pendulum as the subsystem in a coupled dynamical system.

Keywords

chaotic system, perturbed motion, coupled dynamical system, Duffing pendulum

1. Introduction

Nonlinear systems has always been a critical component across numerous domains of human activity, including defense, finance, healthcare, and industrial control systems [1, 2, 3]. In recent decades, the significance of these systems has grown substantially, driven largely by the explosive development of interconnected devices operating under the Internet of Things (IoT) paradigm [4, 5, 6]. The widespread deployment of IoT systems has introduced new challenges in ensuring data confidentiality, integrity, and authenticity, thereby increasing the demand for innovative encryption techniques [7, 8, 9].

Among the various approaches to signals producing and exchanging, nonlinear systems have emerged as a promising and effective tool to produce novel complex signals [10, 11, 12, 13]. These systems offer inherent properties such as sensitivity to initial conditions, ergodicity, and pseudo-random behavior [14, 15, 16, 17, 18, 19]. As a result, the design and analysis of nonlinear systems have become an active area of research, leading to the development of numerous models with distinct dynamic behaviors and applications [20, 21, 22, 23].

One particularly well-studied nonlinear system is the Duffing oscillator, a nonlinear second-order differential equation that models the behavior of certain mechanical and electrical systems [24, 25, 26]. Despite its relatively simple structure, the Duffing oscillator exhibits rich dynamic phenomena, including periodic, quasi-periodic, and chaotic responses [27, 28, 29, 30]. When multiple Duffing oscillators are coupled, the overall system exhibits even more intricate behavior due to the interaction between individual units.

Recent investigations into coupled Duffing pendulums have focused on exploring their complex dynamics through the lens of Hamiltonian chaos, bifurcation theory, and stability analysis [31, 32, 33, 34].

CMSE'25: International Workshop on Computational Methods in Systems Engineering, June 12, 2025, Kyiv, Ukraine
*Corresponding author.
†These authors contributed equally.
volianskyi.roman@lll.kpi.ua (R. Voliansky); ninavolianska@gmail.com (N. Volianska); aji.prasetya.ft@um.ac.id (A. P. Wibawa)
https://epa.kpi.ua/department/staff/roman-voliansky/ (R. Voliansky)
0000-0001-5674-7646 (R. Voliansky); 0000-0002-5996-2341 (N. Volianska); 0000-0002-6653-2697 (A. P. Wibawa)

These studies have uncovered a variety of interesting behaviors such as the emergence of novel motion attractors, fluctuating maximum Lyapunov exponents, phase transitions, and the delineation of stability domains [35, 36]. Such findings not only enhance the understanding of nonlinear coupled systems but also open new avenues for practical applications, including signal generation, control, and system motions' synchronization.

In this work, we further contribute to this area by analyzing the dynamics of coupled Duffing pendulums through the framework of perturbed motion analysis. This perspective allows us to characterize the system's response to small disturbances and identify regions of chaotic and periodic behavior with greater precision. Building upon these insights, we propose a novel design for a chaotic signal generator based on the dynamics of the coupled Duffing pendulum.

2. Method

2.1. Model of the generalized coupled dynamical system in continuous and discrete time domains

Let us consider the generalized second-order nonlinear dynamical system which motion is given by the following normal differential equations

$$\begin{aligned} &\ddot{x}_1 = f_1(\dot{x}_1, x_1, \dot{y}_1, y_1), \\ &x_1(0) = x_{10},\ \dot{x}_1(0) = dx_{10}, \end{aligned} \tag{1}$$

here x_1 is the system generalized coordinates, y_1 is some external excitation signal, x_{10} and dx_{10} are pendulum initial conditions.

We think that the system motions are caused by its initial conditions and some excitation signal y_1 . The signal y_1 is assumed the harmonic one and produced as the result of solution ODE which is similar to (1)

$$\begin{aligned} &\ddot{y}_1 = g_1(\dot{x}_1, x_1, \dot{y}_1, y_1), \\ &y_1(0) = y_{10},\ \dot{y}_1(0) = dy_{10}, \end{aligned} \tag{2}$$

here $g_1(.)$ is some nonlinear function, y_{10} and dy_{10} are exciter initial conditions.

In other words, we consider (2) as the model of some exciter for the system (1). But contrary to the known systems we assume that the dynamic of the exciter is driven by the considered system speed and position. From the mathematical viewpoint, one can consider (1)–(2) as the conjugated equations that define the excited system motion. We call this system the conjugated one. It is clear that the motions of the conjugated system are determined by its parameters, initial conditions, and control signal. External signal y_1 is excluded from consideration by considering the dynamic of the subsystem that it produces. Thus, we claim that the transformation of a dynamical system with external exciter into a conjugated system allows us to take into account the exciter dynamic and exclude from consideration the external excitation signal.

We think that exist another similar system which which we call as the second system and use following equations to define its motions

$$\begin{aligned} &\ddot{x}_2 = f_2(\dot{x}_2, x_2, \dot{y}_2, y_2), \\ &x_2(0) = x_{20},\ \dot{x}_2(0) = dx_{20}; \\ &\ddot{y}_2 = g_2(\dot{x}_2, x_2, \dot{y}_2, y_2), \\ &y_2(0) = y_{20},\ \dot{y}_2(0) = dy_{20}, \end{aligned} \tag{3}$$

here x_2 and y_2 are second conjugated system state variables, $f_2(.)$ and $g_2(.)$ functions which define system motions.

The considered systems interrelate each over in such a way

$$
\begin{aligned}
\ddot{x}_1 &= f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2), \\
\ddot{y}_1 &= g_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2), \\
\ddot{x}_2 &= f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2), \\
\ddot{y}_2 &= g_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2),
\end{aligned}
\tag{4}
$$

here g_{ij} and f_{ij} are functions which allows to take into account the effect of one system to another.

Equations (4) define the dynamic of the coupled system which consists of two interconnected subsystems. The order of such a coupled system equals to sum of orders each subsystem and can becomes quite high while several similar subsystems are being considered.

Since various devices nowadays are designed with discrete-time electronic components and circuits we offer to apply a discrete-time approximations

$$\dot{x} \approx d_1(x, z^{-1}x, T), \tag{5}$$

$$\ddot{x} \approx d_2(x, z^{-1}x, z^{-2}x, T), \tag{6}$$

$$\vdots, \tag{7}$$

$$x^{(n)} \approx d_n(x, z^{-1}x, z^{-2}x, \cdots, z^{-n}x, T), \tag{8}$$

here z^{-i} is a shift operator which takes the value of system state variable x that is defined i time discretization periods T back, and d_i is some approximation procedure to the derivatives in (4) and rewrite them in finite-difference equations form

$$
\begin{aligned}
d_2(x_1, z^{-1}x_1, z^{-2}x_1, T) &= f_{12d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{11}, T), \\
d_2(y_1, z^{-1}y_1, z^{-2}y_1, T) &= g_{12d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{12}, T), \\
d_2(x_2, z^{-1}x_2, z^{-2}x_2, T) &= f_{21d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{21}, T), \\
d_2(y_2, z^{-1}y_2, z^{-2}y_2, T) &= g_{21d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{22}, T),
\end{aligned}
\tag{9}
$$

where f_{ijd} and g_{ijd} are discrete-time images of initial nonlinear functions f_{ij} and g_{ij} and define initial conditions equations as follows

$$
\begin{aligned}
x_1(0) = x_{10},&\ d_1(x_1(0), z^{-1}x_1(0), T) = dx_{10}, \\
y_1(0) = y_{10},&\ d_1(y_1(0), z^{-1}y_1(0), T) = dy_{10}, \\
x_2(0) = x_{20},&\ d_1(x_2(0), z^{-1}x_2(0), T) = dx_{20}, \\
y_2(0) = y_{20},&\ d_1(y_2(0), z^{-1}y_2(0), T) = dy_{20}.
\end{aligned}
\tag{10}
$$

Solution (10) gives us the possibility to define system state variables and the first time moment and the previous time moment

$$
\begin{aligned}
x_1(0) = x_{10},&\ z^{-1}x_1(0) = s_1(x_{10}, dx_{10}, T), \\
y_1(0) = y_{10},&\ z^{-1}y_1(0) = s_1(y_{10}, dy_{10}, T), \\
x_2(0) = x_{20},&\ z^{-1}x_2(0) = s_1(x_{20}, dx_{20}, T), \\
y_2(0) = y_{20},&\ z^{-1}y_2(0) = s_1(y_{20}, dy_{20}, T),
\end{aligned}
\tag{11}
$$

here s_1 means a solution for previous values of i-th state variable. Such a solution makes unambiguous backgrounds to solve the system motion equations (9) which can be given in such a way

$$
\begin{aligned}
x_1 &= f_{s12d}(x_1, z^{-1}x_1, z^{-2}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{11}, T), \\
y_1 &= g_{s12d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, z^{-2}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, u_{12}, T), \\
x_2 &= f_{s21d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, z^{-2}x_2, y_2, z^{-1}y_2, u_{21}, T),
\end{aligned}
\tag{12}
$$

$$y_2 = g_{s21d}(x_1, z^{-1}x_1, y_1, z^{-1}y_1, x_2, z^{-1}x_2, y_2, z^{-1}y_2, z^{-2}y_2, u_{22}, T),$$

where f_{sijd} and g_{sijd} means the solution of the nonlinear equations (9).

It is clear that due to the different subsystem parameters, initial conditions and control signals each subsystem produces various motion trajectories. This fact raises the problem of study the variations subsystem motions.

2.2. Perturbed motion of the generalized coupled system

Let us define the difference between trajectories of the same subsystems as follows

$$\delta x = x_1 - x_2; \tag{13}$$

$$\delta y = y_1 - y_2 \tag{14}$$

and consider the cases of both continuous and discrete time dynamical systems.

We start our studies from the continuous time system and we differentiate (13) to define the first and second derivatives of the trajectory variation

$$\begin{gathered}
\dot{\delta x} = \dot{x}_1 - \dot{x}_2; \\
\dot{\delta y} = \dot{y}_1 - \dot{y}_2; \\
\ddot{\delta x} = \ddot{x}_1 - \ddot{x}_2 = f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2); \\
\ddot{\delta y} = \ddot{y}_1 - \ddot{y}_2 = g_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - g_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2).
\end{gathered} \tag{15}$$

One can consider (15) as differential-algebraic observability equations for coupled dynamical system (4). These equations allows us to define interrelations between the subsystem coordinates x_i, y_i and their variations δx, δy as well as their derivatives. We call trajectory variations and their derivatives as the perturbed motion coordinates. One can find the use of these equations is a quite suitable from control theory viewpoint since it allows defining the derivatives from subsystems trajectory variations without differentiating these variations. At the same time it is clear that to define trajectory variations and their derivatives according to (13) and (15) it is necessary to use the system model (4) as the source of system coordinates.

Another way which allows to exclude the considering of system model while control system is being designed is rewriting (13) and (15) in terms only trajectory variations and their derivatives. The main benefit of such an approach is the possibility to design several dynamical models which are defined with various trajectory variations which defines the structure of control system and its operating algorithm.

The simplest case is the controlling of only one trajectory variations. In this case, the algorithm of perturbed model design can be given in such a way:

- One should select which trajectory variation δx or δy is considered. Let us show the use of proposed algorithm for δx variation.
- Since the system dynamic is defined by eight state variables, it is necessary to differentiate the corresponding trajectory variation for seven times to define the equations to interrelate the system state variables with derivatives of the selecting trajectory variations

$$\delta x = x_1 - x_2; \tag{16}$$

$$\begin{gathered}
\dot{\delta x} = \dot{x}_1 - \dot{x}_2; \\
\ddot{\delta x} = f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2); \\
\delta x^{(3)} = \tfrac{d}{dt} f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - \tfrac{d}{dt} f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) = \\
= \tfrac{df_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2)}{d\dot{x}_1} \ddot{x}_1 + \tfrac{df_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2)}{dx_1} \dot{x}_1 + \cdots \\
\vdots \\
\delta x^{(7)} = \tfrac{d^5}{dt^5} f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - \tfrac{d^5}{dt^5} f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2).
\end{gathered} \tag{17}$$

One can use (16) to define the initial conditions for the perturbed motion by known initial values of the studied dynamical system.

- Solution (16) for the generalized system state variable q can be written down in such a way

$$q_i = s_{\delta x_i}(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(7)}), \tag{18}$$

where $s_{\delta x_i}(.)$ is the nonlinear function which define the interrelations between system state variables and its perturbed motion variables.

- The desired perturbed motion equation can be obtained if one differentiate the last equation (16) and substitute (18) into the defined derivative

$$\delta x^{(8)} = f_8(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(7)}), \tag{19}$$

here $f_8(.)$ is some nonlinear function.

It is clear that the similar transformations can be performed for the case when the perturbed motion is considered for trajectory variation δy. In both cases exact analytical solution of the perturbed motion modeling problem (19) can be obtained only for few very specific cases. That is why we offer to use the numerical methods based on the Newton-Raphson approach or replace the system nonlinearities with their piecewise linear approximations and consider the nonlinear system as a variable-structure one.

Nevertheless, the designed perturbed motion model is defined for only one trajectory variation and allows to control it by solving the minimization problem for the considered trajectory variation. It is clear that the minimization of another variation does not guaranteed. The trying to solve optimization problems for different trajectory variations by using corresponding equation like (19) in the parallel way with using different control inputs can cause control conflicts and unstudied system dynamic.

That is why for the case when it is necessary to minimize both system trajectory variations we offer another approach to model the system perturbed motion.

In this case we modify the above-given algorithm to design the perturbed motion in terms of both perturbed motion coordinates:

- Both trajectory variations (13) are considered.
- Each of these variations are differentiated for three times to define the interrelations between the perturbed coordinates and system motion coordinates

$$\begin{aligned}
\delta x &= x_1 - x_2; \\
\delta y &= y_1 - y_2; \\
\dot{\delta x} &= \dot{x}_1 - \dot{x}_2; \\
\dot{\delta y} &= \dot{y}_1 - \dot{y}_2; \\
\ddot{\delta x} &= \ddot{x}_1 - \ddot{x}_2 = f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2); \\
\ddot{\delta y} &= \ddot{y}_1 - \ddot{y}_2 = g_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - g_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2); \\
\delta x^{(3)} &= \tfrac{d}{dt} f_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - \tfrac{d}{dt} f_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2); \\
\delta y^{(3)} &= \tfrac{d}{dt} g_{12}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2) - \tfrac{d}{dt} g_{21}(\dot{x}_1, x_1, \dot{y}_1, y_1, \dot{x}_2, x_2, \dot{y}_2, y_2).
\end{aligned} \tag{20}$$

- Solution (13) for system state variables allows us to define them in terms of both perturbed motion coordinates and their derivatives. The generalized solution (18)

$$q_i = s_{\delta x_i}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(3)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}), \tag{21}$$

- The desired perturbed motion equations are defined by differentiating the two last equations in (20) and substituting (21) into these derivatives

$$\begin{aligned}
\delta x^{(4)} &= f_4(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(3)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}); \\
\delta y^{(4)} &= g_4(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(3)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}),
\end{aligned} \tag{22}$$

here $f_4(.)$ and $g_4(.)$ are some functions.

As one can see from comparison of (19) and (22) the order of perturbed motion equations of the considered system does not depend on the way to represent its motion and it equals to 8 which equals the order of initial coupled system (4). One can use this fact to check the correctness of performed transformations while the perturbed motion model for the specific coupled dynamical system is being designed.

One can easy transform (19) into discrete-time domain by applying to them approximations (5) and representing the system dynamic in the explicit form as follows

$$\begin{aligned}\delta x &= f_{sd8}(z^{-1}\delta x, z^{-2}\delta x, \cdots, z^{-7}\delta x, u_{11}, \cdots, z^{-6}u_{11}, u_{21}, \cdots, z^{-6}u_{21}), \\ \delta y &= g_{sd8}(z^{-1}\delta y, z^{-2}\delta y, \cdots, z^{-7}\delta y, u_{12}, \cdots, z^{-6}u_{12}, u_{22}, \cdots, z^{-6}u_{22}),\end{aligned} \tag{23}$$

where $f_{sd8}(.)$ and $g_{sd8}(.)$ are discrete-time images of $f_8(.)$ and $g_8(.)$ functions which are used to define the perturbed motion for δx and δy perturbed motion coordinates.

In similar way (22) can be rewriting

$$\begin{aligned}\delta x = f_{sd4}(&z^{-1}\delta x, z^{-2}\delta x, \cdots, z^{-4}\delta x, z^{-1}\delta y, z^{-2}\delta y, \cdots, z^{-4}\delta y, \\ &u_{11}, z^{-1}u_{11}, u_{21}, z^{-1}u_{21}, u_{12}, z^{-1}u_{12}, u_{22}, z^{-1}u_{22}); \\ \delta y = g_{sd4}(&z^{-1}\delta x, z^{-2}\delta x, \cdots, z^{-4}\delta x, z^{-1}\delta y, z^{-2}\delta y, \cdots, z^{-4}\delta y, \\ &u_{11}, z^{-1}u_{11}, u_{21}, z^{-1}u_{21}, u_{12}, z^{-1}u_{12}, u_{22}, z^{-1}u_{22}),\end{aligned} \tag{24}$$

where $f_{sd4}(.)$ and $g_{sd4}(.)$ are discrete-time images of $f_4(.)$ and $g_4(.)$ functions which are used to define the perturbed motion for δx and δy perturbed motion coordinates.

Analysis of the discrete-time perturbed motion equations (23) and (24) allows us to claim the dependency of these equations from the previous values of perturbed motion coordinates. So, one should take into account this fact and reserve memory to save this data while the considered systems are implemented with various digital devices.

Generally speaking, the above designed continuous and discrete-time models to study the systems perturbed motions can be used as the sources of some signals which differ from the initial outputs of each subsystem. It can be very useful in the case when the considered system has chaotic dynamic. In this case the use of the proposed approach allows us to solve the direct dynamic problem and define system motions by known control signals and initial conditions.

2.3. Synchronization of the coupled subsystems

However, the main benefit of the perturbed motion equations can be found while one design the control system. The design of such a system can be considered from a mathematical viewpoint as the problem of minimizing the perturbed motion trajectories which means the coincidence of both subsystems motions.

We consider the system optimization problem as the solution of inverse dynamic problem which means to determine the control signals by known motions. We assume that system has at least one non-zero component of initial condition vector and we define such a control signals which make the perturbed system motion asymptotically stable and tend it to zero.

It is a well known fact that the simplest asymptotic stable dynamical system is the linear first-order one which motion can be given following linear differential operator

$$W(s) = \frac{1}{Ts+1}, \tag{25}$$

here T is a lag time of a desired system and s is a Laplace operator.

Such an operator defines two components of the closed-loop system: feedback with gain equals to one and integrator in feedforward channel with $1/T$ gain. One can use the last fact to claim that the operator, which define the controller structure and parameters, should compensate the system inner dynamic and define the desired motion of the open-loop system as follows

$$\dot{q}_i = \frac{1}{T}v_j, \tag{26}$$

here q_i and v_j are some generalized output variable and control signal.

It is clear that compensation of system inner dynamic can be performed if one solves the inverse dynamic problem for the system. From the mathematical viewpoint such a solution means the use of previously-written perturbed motions to define the control signal as function of the perturbed coordinates. Since the considered dynamical system is multichannel one can use the same perturbed motion equation to define control signals for various channels. Thus, if one takes into account (19) it becomes possible to define following control signals

$$\begin{aligned} u_{11} &= f_{8u11}(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(8)}, u_{21}, \cdots, u_{21}^{(6)}); \\ u_{21} &= f_{8u21}(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(8)}, u_{11}, \cdots, u_{11}^{(6)}). \end{aligned} \tag{27}$$

It should be mentioned that both of control signals allows to compensate inner system perturbed motion and in-joint to (26) gives the possibility to form the desired motion which is defined by (25)

$$\begin{aligned} u_{11} &= \frac{1}{T}\int f_{8u11}(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(8)}, u_{21}, \cdots, u_{21}^{(6)})dt; \\ u_{21} &= \frac{1}{T}\int f_{8u21}(\delta x, \dot{\delta x}, \ddot{\delta x}, \cdots, \delta x^{(8)}, u_{11}, \cdots, u_{11}^{(6)})dt. \end{aligned} \tag{28}$$

These control signal can be supplied to the system in separate way to build a single-channel control system on in parallel to design the dual-channel one.

The number of control signals which can be defined as the solution of inverse dynamic problem increases for the case when system perturbed motion is defined by (22). Generally speaking, due to the coupled nature of the considered system as well as dependency of each equation in (22) from all control inputs one can use any of them to control any perturbed motion coordinate. The most specific case is the use only one equations from (22) to define all control signals which allows to minimize the only one perturbed motion coordinate. The following equations allows us to compensate inner perturbed motion dynamic for the channel of δx

$$\begin{aligned} u_{11} &= f_{4u11}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{21}, \dot{u}_{21}, u_{12}, \dot{u}_{12}, u_{22}, \dot{u}_{22}); \\ u_{12} &= f_{4u12}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{21}, \dot{u}_{21}, u_{11}, \dot{u}_{11}, u_{22}, \dot{u}_{22}); \\ u_{21} &= f_{4u21}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{11}, \dot{u}_{11}, u_{12}, \dot{u}_{12}, u_{22}, \dot{u}_{22}); \\ u_{22} &= f_{4u22}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{21}, \dot{u}_{21}, u_{12}, \dot{u}_{12}, u_{11}, \dot{u}_{11}). \end{aligned} \tag{29}$$

The use (29) to implement the control system cause the necessity to consider the four-channel controller to control δx variation. It is clear that the coordinate δy in this case is non-controlled and it can take any values.

We offer avoid the uncontrolled system dynamic by using (22) to define the control signals for each control channel. Here we take into account the place where the signal is supplied to the system and we use the control signals in the same subsystems to compensate their inner perturbed motions

$$\begin{aligned} u_{11} &= f_{4u11}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{21}, \dot{u}_{21}, u_{12}, \dot{u}_{12}, u_{22}, \dot{u}_{22}); \\ u_{21} &= f_{4u21}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(4)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(3)}, u_{11}, \dot{u}_{11}, u_{12}, \dot{u}_{12}, u_{22}, \dot{u}_{22}); \\ u_{12} &= f_{4u12}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(3)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(4)}, u_{21}, \dot{u}_{21}, u_{11}, \dot{u}_{11}, u_{22}, \dot{u}_{22}); \\ u_{22} &= f_{4u22}(\delta x, \dot{\delta x}, \ddot{\delta x}, \delta x^{(3)}, \delta y, \dot{\delta y}, \ddot{\delta y}, \delta y^{(4)}, u_{21}, \dot{u}_{21}, u_{12}, \dot{u}_{12}, u_{11}, \dot{u}_{11}). \end{aligned} \tag{30}$$

The designed in such a way control system can be considered as the multi-loop multi-channel control system and each channel and loop are controlled in a parallel way. It is necessary to say that the similar control signals can be defined in the discrete-time by using (23) and (24).

We show the use of our approach by designing a control system for coupled Duffing pendulum.

3. Results and discussion

3.1. Coupled Duffing pendulum with driven exciter

We show the use of our approach by studying the perturbed motion of the coupled Duffing pendulum. It is a well known fact that the single Duffing pendulum's dynamic can given as follows

$$\begin{aligned} &\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma cos(\omega t), \\ &x(0) = x_0,\ \dot{x}(0) = dx_0, \end{aligned} \tag{31}$$

here x is a pendulum position, α,β,δ are pendulum parameters, and γ, ω are parameters of external excitation signal, x_0 and dx_0 are pendulum initial position and speed.

One can rewrite (31) in form of conjugated equations by considering an exciter dynamic

$$\begin{aligned} &\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma y; \\ &\ddot{y} + \omega^2 y = 0, \\ &x(0) = x_0,\ \dot{x}(0) = dx_0,\ y(0) = y_0, \dot{y}(0) = dy_0, \end{aligned} \tag{32}$$

here y is an exciter output, y_0 and dy_0 are exciter initial position and speed.

Results of numerical solution (32) with the simplest Euler method are shown in Figure 1 and Figure 2

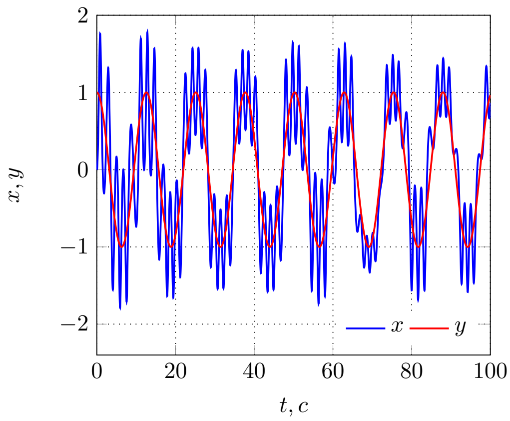

Figure 1: Oscillations in Duffing pendulum

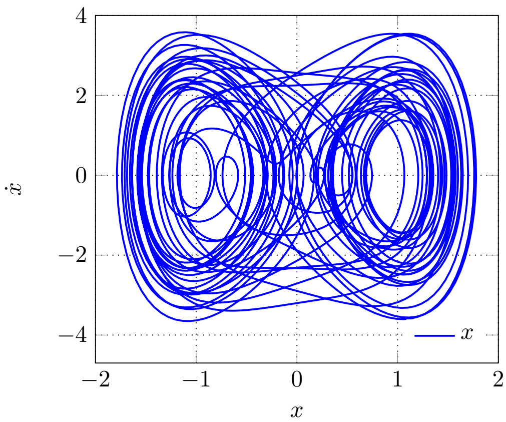

Figure 2: Attractor of Duffing pendulum

Here and further we use following pendulum parameters $\alpha = 1$, $\beta = 5$, $\gamma = 8$, $\delta = 0.02$, and $\omega = 0.5$. This results are the similar to a well-known one and prove correctness of the designed model.

It is clear that the pendulum position depend on the exciter output. This position can be more complex and upredictive in case of the driven exciter (Figure 3–Figure 4)

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma y;\ \ddot{y} + \omega^2 y + \epsilon x = 0. \tag{33}$$

These simulation results are obtained for $\epsilon = 0.6$. As one can see from Figure 3 and Figure 4 the use of driven exciter which output depend on the pendulum position allows us to form non-regular excitation signal y which dramatically changes the pendulum oscillations and deforms the pendulum attractor.

We consider the above-studied driven Duffing pendulum as one of two coupled Duffing pendulums which dynamic is defined as follows

$$\begin{gathered} \ddot{x_1} + \delta_{11}\dot{x_1} + \alpha_{11}x_1 + \beta_{11}x_1^3 + \delta_{12}\dot{x_2} + \alpha_{12}x_2 + \beta_{12}x_2^3 = \gamma_{11}y_1 + \gamma_{12}y_2; \\ \ddot{y_1} + \omega_{11}^2 y_1 + \omega_{12}^2 y_2 + \epsilon_{11}x_1 + \epsilon_{12}x_2 = 0; \\ \ddot{x_2} + \delta_{21}\dot{x_1} + \alpha_{21}x_1 + \beta_{21}x_1^3 + \delta_{22}\dot{x_2} + \alpha_{22}x_2 + \beta_{22}x_2^3 = \gamma_{21}y_1 + \gamma_{22}y_2; \\ \ddot{y_2} + \omega_{21}^2 y_1 + \omega_{22}^2 y_2 + \epsilon_{21}x_1 + \epsilon_{22}x_2 = 0, \end{gathered} \tag{34}$$

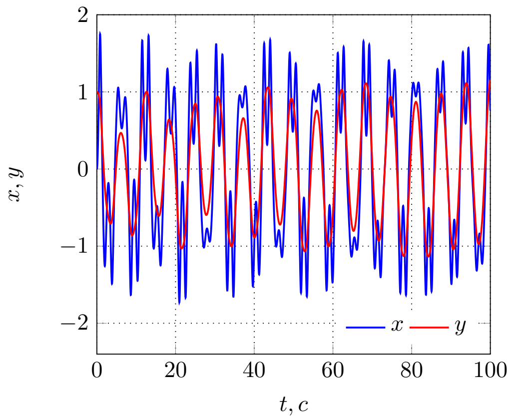

Figure 3: Oscillations in Duffing pendulum with driven exciter

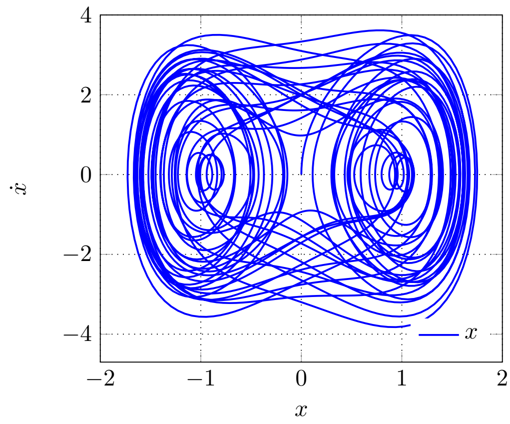

Figure 4: Attractor of Duffing pendulum with driven exciter

here indices i and j means the effect of i pendulum on j one.

From the physically implementation viewpoint the above-defined interconnections means the use spring with a nonlinear stiffness and internal dumping to connect both pendulums.

Simulation results are given in Figure 5–Figure 8 for the following pendulum parameters $\alpha_{11} = 1$, $\alpha_{12} = 0.8$, $\alpha_{21} = 0.9$, $\alpha_{22} = 1.1$, $\beta_{11} = 5$, $\beta_{12} = 4$, $\beta_{21} = 4.5$, $\beta_{22} = 5.5$, $\delta_{11} = 0.02$, $\delta_{12} = 0.015$, $\delta_{21} = 0.01$, $\delta_{22} = 0.03$, $\gamma_{11} = 8$, $\gamma_{12} = 9$, $\gamma_{22} = 9$, $\gamma_{21} = 7$, $\omega_{11} = 0.5$, $\omega_{12} = 0.4$, $\omega_{22} = 0.8$, and $\omega_{21} = 0.2$.

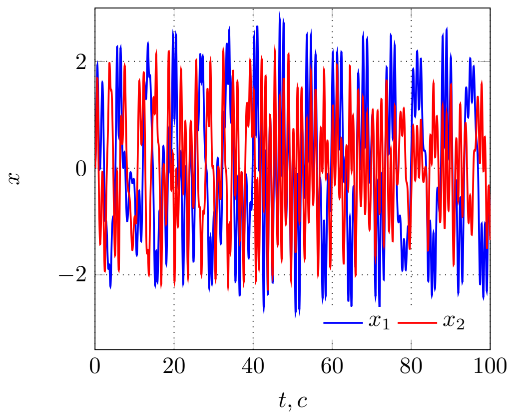

Figure 5: Oscillations in coupled Duffing pendulum

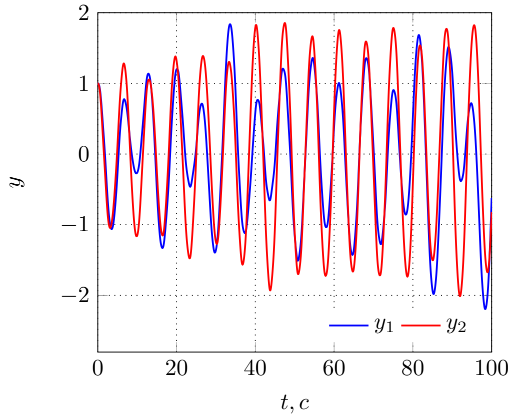

Figure 6: Coupled exciters outputs

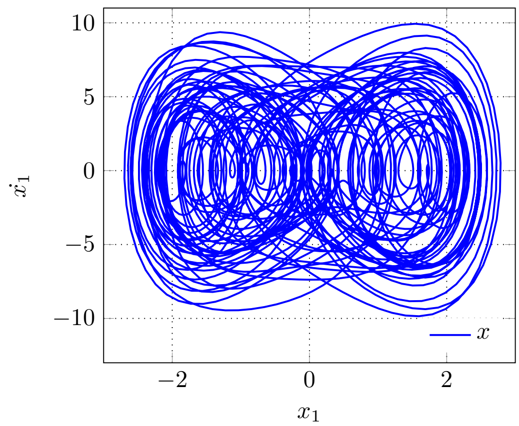

Figure 7: Attractor in coupled Duffing pendulum

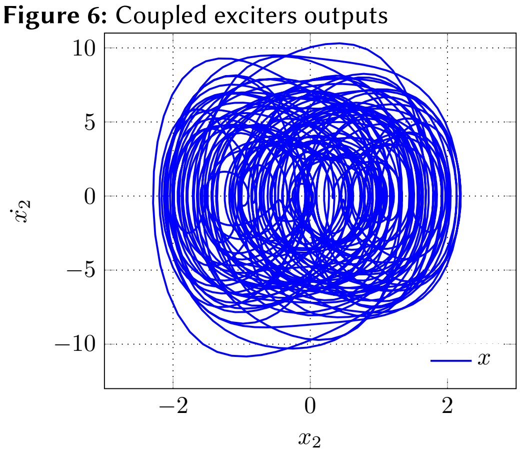

Figure 8: Attractor in coupled Duffing pendulum

As one can see the dynamic of two coupled chaotic system differ from the dynamic of one system.

One can use this fact to design the novel chaotic systems by coupling the known ones.

3.2. Perturbed motion model of coupled Duffing pendulums

The way to design a novel chaotic system is use of combnations of state variables of known systems. One of such combinations which has a physical meaning is the system perturbed motion which can be considered as pendulums and exciters trajectory variations

$$\begin{aligned} \delta x &= x_1 - x_2; \\ \delta y &= y_1 - y_2; \\ \dot{\delta x} &= \dot{x_1} - \dot{x_2}; \\ \dot{\delta y} &= \dot{y_1} - \dot{y_2}. \end{aligned} \tag{35}$$

In this case (35) can be considered as the observability equations for (34) and both of these equations make state space equations of the perturbed motion of coupled Duffing pendulums. Numerical solution such a system is shown in Figure 9–12.

As one can see the considering of pendulum's perturbed motions allows us to define the novel chaotic system which motions and attractors differ from the initial ones. Moreover, analysis of curves given in Figure 9–12 shows that the designed in such a way system does not have two equilibrium points which have the initial classical Duffing pendulum.

It is clear that the considered system can be easy implemented by using various digital devices like MCU or FPGA. Such an implementation should be based on the use of various approximations of derivative operator to solve (34) in a numerical way. In our paper we consider the implementation of developed models in Arduino Due board by using backward finite-difference approximation which is defined as follows

$$\frac{d}{dt} \approx \frac{1 - z^{-1}}{T z^{-1}}, \tag{36}$$

here T is a discretization time and z^{-1} is a shift operator which define the previous value of the considered system's state variables.

The main feature of such an approach is the necessity to solve at first the differential equations of the pendulum motions and then use the observability equations to define its perturbed motions. One can find it is not very convenient to use such equations to various control problems such as a solution of inverse dynamic problem. That is why we offer to rewrite (34) and (35) in form of differential equations only.

It is clearly understood that such rewriting requires multiply differentiating of the perturbed motion coordinates δx and δy which components are defined by using the nonlinear differential equations. Since such a differentiating can cause the determination of derivatives for pendulum state variables with highly nonlinear equations which cannot be solved in an analytical way, we offer to replace the pendulum cubic nonlinearity with piece-wise linear function as follows

$$x^3 \approx \begin{cases} k_1 x + b_1 & if \quad x_0 \le x < x_1; \\ k_2 x + b_2 & if \quad x_1 \le x < x_2; \\ & \vdots \\ k_n x + b_n & if \quad x_{n-1} \le x < x_n, \end{cases} \tag{37}$$

where k_i and b_i are piece-wise linear approximation factors and x_i is coordinate of fracture point, and n is number of these points.

The use of approximation (37) allows us to rewrite (34) in linear-like matrix form

$$\dot{\mathbf{Q}} = \mathbf{AQ} + \mathbf{B}, \tag{38}$$

$$\mathbf{Q} = \begin{pmatrix} x_1 & \dot{x_1} & y_1 & \dot{y_1} & x_2 & \dot{x_2} & y_2 & \dot{y_2} \end{pmatrix}^T, \tag{39}$$

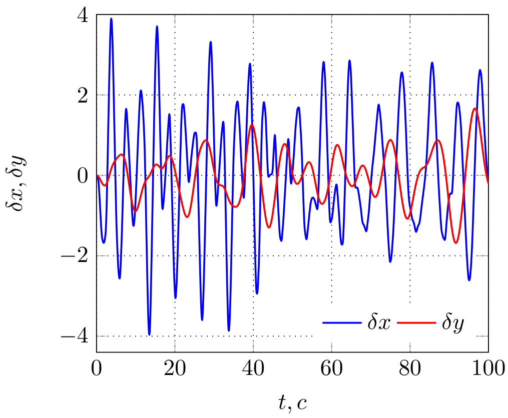

Figure 9: Pendulum's positions variations

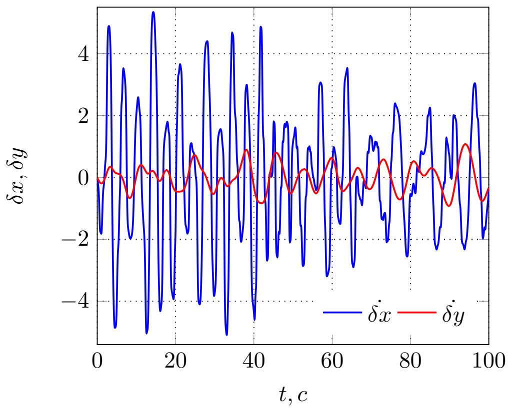

Figure 10: Pendulum's speed variations

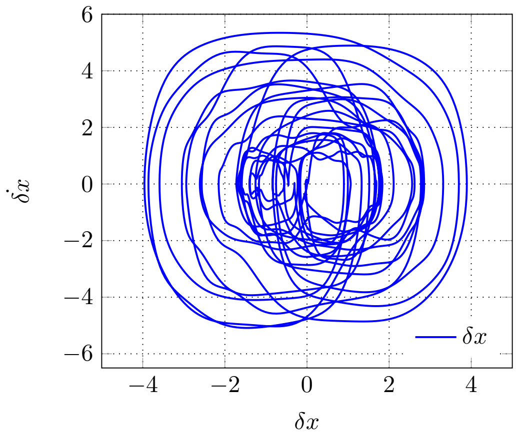

Figure 11: The perturbed motion's pendulum attractor

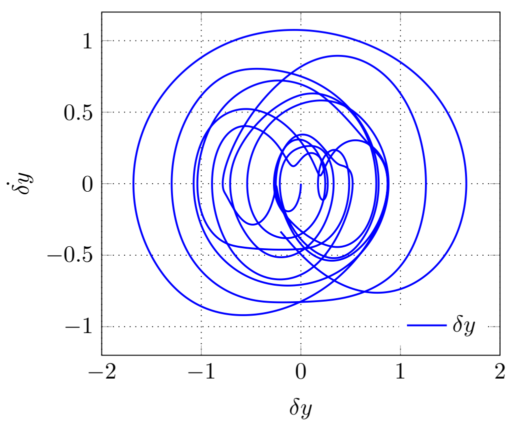

Figure 12: The perturbed motion's exciter attractor

$$\mathbf{B} = \begin{pmatrix} 0 & -(\beta_{11}+\beta_{12})b_i & 0 & 0 & 0 & (\alpha_{22}+\beta_{22}k_i) & 0 & 0 \end{pmatrix}^T.$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(\alpha_{11}+\beta_{11}k_i) & -\delta_{11} & -(\alpha_{12}+\beta_{12}k_i) & -\delta_{12} & \gamma_{11} & 0 & \gamma_{12} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\epsilon_{11} & 0 & -\epsilon_{12} & 0 & -\omega_{11}^2 & 0 & -\omega_{12}^2 y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -(\alpha_{21}+\beta_{21}k_i) & -\delta_{21} & -(\alpha_{22}+\beta_{22}k_i) & -\delta_{22} & \gamma_{21} & 0 & \gamma_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\epsilon_{21} & 0 & -\epsilon_{22} & 0 & -\omega_{21}^2 & 0 & -\omega_{22}^2 & 0 \end{pmatrix}. \tag{40}$$

$$\ddot{x_1} = -\delta_{11}\dot{x_1} - (\alpha_{11}+\beta_{11}k_i)x_1 - \delta_{12}\dot{x_2} - (\alpha_{12}+\beta_{12}k_i)x_2 + \gamma_{11}y_1 + \gamma_{12}y_2 - (\beta_{11}+\beta_{12})b_i; \tag{41}$$
$$\ddot{y_1} = -\omega_{11}^2 y_1 - \omega_{12}^2 y_2 - \epsilon_{11}x_1 - \epsilon_{12}x_2;$$
$$\ddot{x_2} = -\delta_{21}\dot{x_1} - (\alpha_{21}+\beta_{21}k_i)x_1 - \delta_{22}\dot{x_2} - (\alpha_{22}+\beta_{22}k_i)x_2 + \gamma_{21}y_1 + \gamma_{22}y_2 - (\beta_{21}-\beta_{22})b_i;$$
$$\ddot{y_2} - \omega_{21}^2 y_1 - \omega_{22}^2 y_2 - \epsilon_{21}x_1 - \epsilon_{22}x_2,$$

The perturbed motion system's output we define by matrix observability equation

$$\mathbf{\Delta} = \mathbf{CQ}, \tag{42}$$

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} \end{pmatrix}, \tag{43}$$

where c_{ij} matrix components take values ± 1 which depend on sign near state variable in (35).

Here we show the most general case for the matrix $\mathbf{C}$ when perturbed motion can be considered for two variations at the same time. In more specific case when only one variation of pendulum state variables are studied one should reduce a number of matrix $\mathbf{C}$ rows to one. Thus, if one study the perturbed motion of the pendulums positions (43) can be given as follows

$$\mathbf{C_p} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

for the case of studying the perturbed motions of pendulums exciters

$$\mathbf{C_e} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \tag{45}$$

and if both perturbed motions are studied

$$\mathbf{C_{pe}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}. \tag{46}$$

The usage of matrix equations (38) and (42) gives us the possibility to write down equations which interrelate perturbed motion coordinates and their derivatives with the pendulum state variables

$$\mathbf{Y} = \mathbf{WQ} + \mathbf{V}, \tag{47}$$

$$\mathbf{Y} = \begin{pmatrix} \boldsymbol{\Delta} & \dot{\boldsymbol{\Delta}} & \ddot{\boldsymbol{\Delta}} & \cdots & \boldsymbol{\Delta}^{(\mathbf{7})} \end{pmatrix}^T; \tag{48}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{C} & \mathbf{AC} & \mathbf{A^2C} & \cdots & \mathbf{A^7C} \end{pmatrix}^T,$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{C} & \mathbf{BC} & \mathbf{B^2C} & \cdots & \mathbf{B^7C} \end{pmatrix}^T.$$

Solution (44) allows us to define state space variables of coupled pendulums with perturbed motion coordinates

$$\mathbf{Q} = \mathbf{W^{-1}Y}, \tag{49}$$

The final perturbed motion equation can be given as follows

$$\boldsymbol{\Delta}^{(8)} = \mathbf{A^8CW^{-1}Y} + \mathbf{B^8C}. \tag{50}$$

The main feature of (46) is its dependence only perturbed motions coordinates and pendulums parameters. Numerical solution (46) allows us to get the results which are similar to shown in Figure.9–Figure.12 but only for one perturbed motion coordinate δx or δy.

If one studies the perturbed motions of pendulums and exciters at the same time, he can should modify the above-given formulas in such a way

$$\mathbf{Y} = \begin{pmatrix} \boldsymbol{\Delta}_\mathbf{x} & \dot{\boldsymbol{\Delta}}_\mathbf{x} & \ddot{\boldsymbol{\Delta}}_\mathbf{x} & \cdots & \boldsymbol{\Delta}_\mathbf{x}^{(\mathbf{3})} \\ \boldsymbol{\Delta}_\mathbf{y} & \dot{\boldsymbol{\Delta}}_\mathbf{y} & \ddot{\boldsymbol{\Delta}}_\mathbf{y} & \cdots & \boldsymbol{\Delta}_\mathbf{y}^{(\mathbf{3})} \end{pmatrix}^T; \tag{51}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{C} & \mathbf{AC} & \mathbf{A^2C} & \cdots & \mathbf{A^3C} \end{pmatrix}^T,$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{C} & \mathbf{BC} & \mathbf{B^2C} & \cdots & \mathbf{B^3C} \end{pmatrix}^T$$

and

$$\boldsymbol{\Delta}^{(4)} = \mathbf{A^4CW^{-1}Y} + \mathbf{B^4C}. \tag{52}$$

Defined in such a way perturbed motion models allows to study trajectory variations for coupled Duffing pendulums without solution of each pendulum equations. Since they are written down by using piece-wise linear functions one should use (45) while piece-wise linear factors are defined.

4. Conclusions

The considering some dynamical system as the part of coupled system allows us to change the initial system's dynamic and design the system with unique features and characteristics. These characteristics can be improved by considering the trajectory variations for each subsytem. Using these trajectory variations as novel state variables allows designing novel dynamical system based on the known one. The order of novel system equals to initial one. The designed in such a way dynamical systems in various applications.

We see the future development of our work in transforming the proposed approach in discrete-time domain to design and study the controlled system motions as well as to solve the inverse dynamic problem to define an external signal by known system coordinates.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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